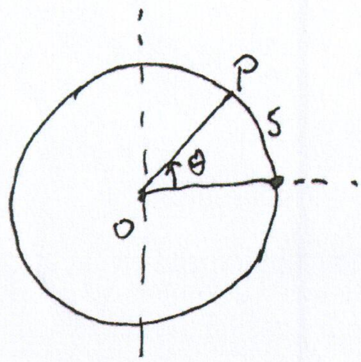
Chap 10. Rotation of a rigid Object about a fixed Axis

Rigid ≡ Non deformable

10-1 Angular position, velocity and Acceleration



$s = r\theta$

$\theta = \frac{s}{r}$ [radian]

1 Radian ≡ the angle subtended by an arc length of the radius

$1 \text{ rad} = \frac{360°}{2\pi} \sim 57.3°$

$\Delta\theta \equiv \theta_f - \theta_i$

Angular speed

$$\bar{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

Angular speed

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular acceleration

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

for a rigid object rotates about a fixed axis, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and acceleration.

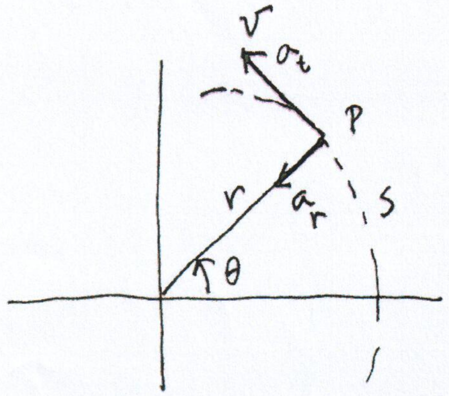
Use right hand rule to decide the directions

10-2. Rotation Kinematics : Rotational motion with constant angular acceleration

$$d\omega = \alpha\, dt$$

$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_0^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Relating angular and linear quantities



$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\therefore \underline{\underline{v = r\omega}}$$

$$a_t = \frac{dv}{dt} = \frac{r\,d\omega}{dt} = r\alpha$$

$$a_r = \frac{v^2}{r} = r\omega^2$$

$$\vec{a} = \vec{a}_t + \vec{a}_r \Rightarrow |a| = \sqrt{a_r^2 + a_t^2}$$

$$|a| = \sqrt{r^2\alpha^2 + r^2\omega^4}$$
$$= r\sqrt{\alpha^2 + \omega^2}$$

10.4 Rotational Kinetic energy

$K_i = \frac{1}{2}m_i v_i^2$ for ith point with mass m_i

$$KR = \sum_i K_i = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i r_i^2 \omega^2$$
$$= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \qquad I \equiv \sum_i m_i r_i^2$$

= moment of inertia

$$= \frac{1}{2}I\omega^2$$ — Rotational Kinetic energy.

10.5 Calculating moments of inertia

$$I = \sum_i r_i^2 m_i$$

$$= \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i$$

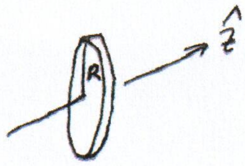
$$= \int r^2 dm \qquad \text{But } dm = \rho\, dV$$

$$I = \int \rho r^2 dV$$

Example of the moment of inertia

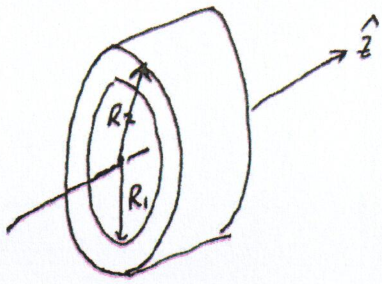
Page 227. Rotational inertias

(a) Hoop about central axis



$$I = \int_0^M R^2\,dm = MR^2$$

(b) Annular cylinder



$$I = \int dm\, r^2$$

$$dm = 2\pi r H \rho\, dr$$

$$= \int 2\pi r H \rho\, dr\, r^2$$

$$= 2\pi H \rho \int_{R_1}^{R_2} r^3 dr = 2\pi H \rho \tfrac{1}{4}(R_2^4 - R_1^4)$$

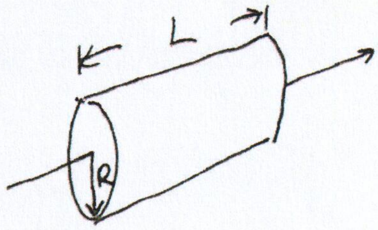
But $M = \pi R_2^2 H\rho - \pi R_1^2 H \rho$

$= \pi H \rho (R_2^2 - R_1^2)$

$$\therefore I = \frac{\pi}{2} H\rho (R_2^4 - R_1^4) = \frac{\pi}{2} H\rho (R_2^2 + R_1^2)(R_2^2 - R_1^2)$$

$$= \tfrac{1}{2} M (R_1^2 + R_2^2)$$

(c) Solid cylinder



$$I = \int r^2 dm$$

$$= \int r^2 2\pi r\, dr\, L\rho$$

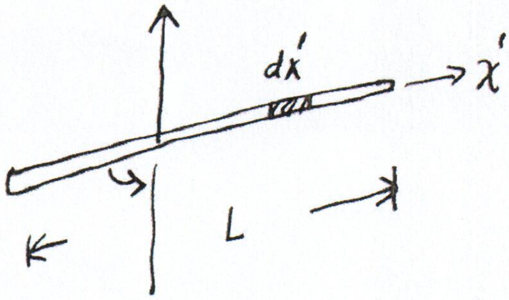
$$= 2\pi L \rho \int_0^R r^3 dr$$

$$= 2\pi L \rho \frac{R^4}{4}$$

$$= \tfrac{1}{2} M R^2$$

(d) Thin rod about axis perpendicular to the length

$$dm = \lambda\, dx' = \frac{M}{L} dx'$$



$$I = \int r^2 dm = \int_{-\frac{L}{2}}^{+L/2} \lambda (x')^2 dx'$$

$$= \frac{M}{L} \int_{-L/2}^{L/2} (x')^2 dx'$$

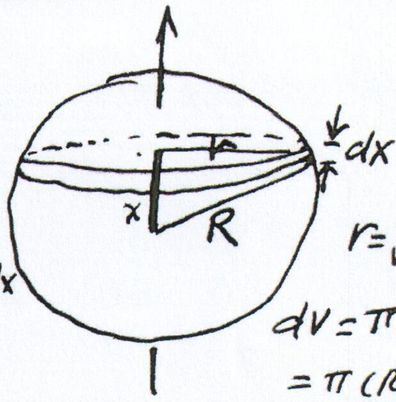
$$= \frac{M}{L} \left[\frac{(x')^3}{3} \right]_{-L/2}^{L/2}$$

$$= \frac{1}{12} M L^2$$

(e) Solid sphere about any diameter

cut the sphere into many horizontal disks of radius r.

$$dI_{disk} = \frac{1}{2} r^2 dm$$



$r = \sqrt{R^2 - x^2}$

$dV = \pi r^2 dx$
$= \pi (R^2 - x^2) dx$

$dm = \rho\, dV = \pi \rho (R^2 - x^2) dx$

$$dI_{disk} = \frac{1}{2} r^2 dm = \frac{1}{2} (R^2 - x^2)\, \pi \rho (R^2 - x^2) dx$$

$$= \frac{\pi \rho}{2} (R^2 - x^2)^2 dx$$

$$I = (2) \frac{\pi \rho}{2} \int_0^R (R^2 - x^2) dx$$

$$= \frac{8 \pi \rho R^5}{15}$$

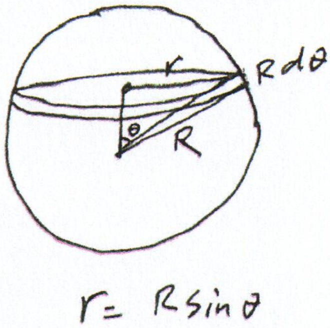
But $\rho = \frac{M}{V} = \frac{3M}{4 \pi R^3}$

$$\therefore I = \left(\frac{8 \pi R^5}{15} \right) \left(\frac{3M}{4 \pi R^3} \right) = \frac{2}{5} M R^2$$

Note: There are many different ways to calculate the moment of inertia; this is just one way.

(g) Thin spherical shell about any diameter

$$I = \int_0^{\pi} r^2\, dm$$



$$= \int_0^{\pi} R^2 \sin^2\theta \cdot 2\pi R^2 \sin\theta\, d\theta$$

$$= 2\pi R^4 \int_0^{\pi} \sin^2\theta \sin\theta\, d\theta$$

$$= 2\pi R^4 \int_{+1}^{-1} (1-\cos^2\theta)\, d(-\cos\theta)$$

$$= 2\pi R^4 \int_{-1}^{+1} (1-\cos^2\theta)\, d(\cos\theta)$$

$$= 2\pi R^4 \left[x - \frac{1}{3}x^3 \right]_{-1}^{1}$$

$$= 2\pi R^4 \cdot \frac{4}{3}$$

$$= \frac{8}{3}\pi R^4 = \frac{2}{3} M R^2$$

$r = R\sin\theta$

$dm = 2\pi r\, R\, d\theta$

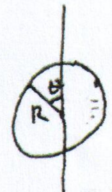
$= 2\pi R \sin\theta\, R\, d\theta$

$= 2\pi R^2 \sin\theta\, d\theta$

$M = 4\pi R^2$

Assume σ = density = 1

(h) Hoop about any diameter



$$I = \int R^2 dm = 2\int_0^{\pi} (r\sin\theta)^2 dm$$

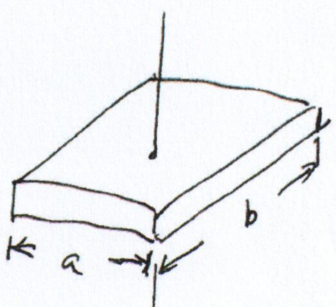
$$= 2\int_0^{\pi} R^2 \sin^2\theta\, R\,d\theta = 2R^3 \int_0^{\pi} \sin^2\theta\, d\theta$$

~~$= \int_0^{\pi} R^2 R\,d\theta = \pi R^3 = \frac{1}{2} MR^2$~~. $M = 2\pi R$ $= \frac{2}{2} R^3 \pi$

$= \frac{1}{2} M R^2$

(i) Slab about perpendicular axis through center

$$I = \int r^2 dm \quad = I_x + I_y$$

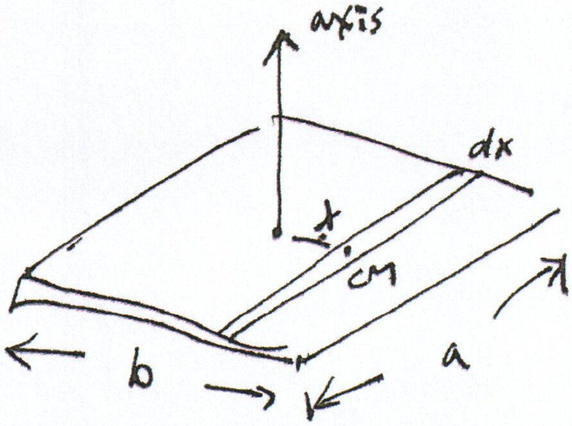


$$= \int (x^2+y^2)\, dm$$

$$= \int x^2 dm + \int y^2 dm$$

$$= \frac{1}{12}(a^2+b^2)$$

Also see page P10-6-1



$dm = a\,dx$ Assume density $= \sigma$

$$\frac{dM}{M} = \frac{a\,dx}{ab} \Rightarrow dm = \frac{M}{b}dx$$

$$dI \text{ of this strip} = dI_{cm} + dm\,h^2$$

$$= \frac{1}{12}(dm)\,a^2 + dm\,x^2$$

$$= \frac{1}{12}\left(\frac{M}{b}\right)dx\,a^2 + \left(\frac{M}{b}\right)dx\,x^2$$

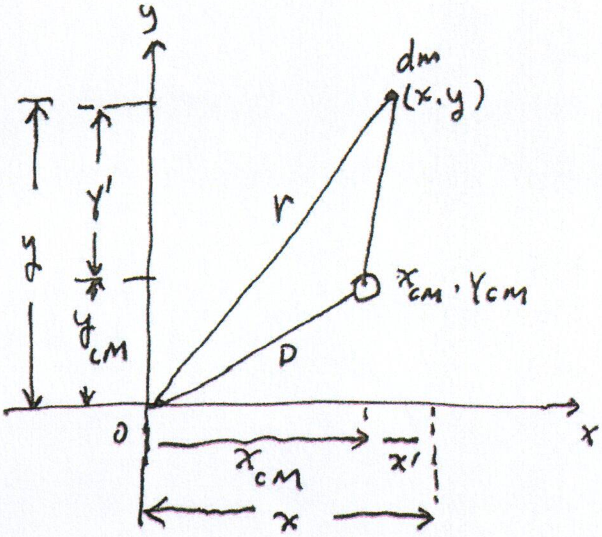
$$= \frac{Ma^2}{12b}dx + \frac{M}{b}x^2dx$$

$$I = \int dI = \frac{Ma^2}{12b}\int_{-b/2}^{b/2}dx + \frac{M}{b}\int_{-b/2}^{b/2}x^2dx$$

$$= \frac{1}{12}M(a^2+b^2)$$

Parallel axis theorem

$$I = I_{cm} + MD^2$$



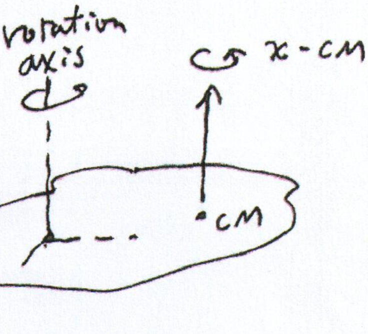
$$I = \int r^2 dm$$

$$= \int (x^2 + y^2) dm$$

But $x = x' + x_{cm}$

$y = y' + y_{cm}$

$$\therefore I = \int [(x' + x_{cm})^2 + (y' + y_{cm})^2] dm$$



$$= \int (x'^2 + 2x'x_{cm} + x_{cm}^2 + y'^2 + 2y'y_{cm} + y_{cm}^2) dm$$

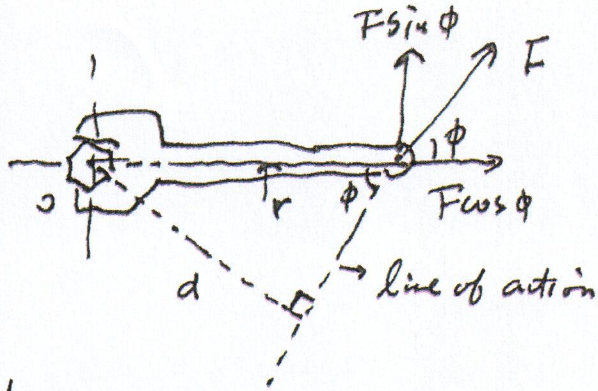
$$= \underbrace{\int (x'^2 + y'^2) dm}_{I_{cm}} + \underbrace{2x_{cm} \int x' dm}_{=0} + \underbrace{2y_{cm} \int y' dm}_{=0} + \underbrace{(x_{cm}^2 + y_{cm}^2) \int dm}_{D^2 M}$$

definition of center of mass

$$\therefore I = I_{cm} + MD^2$$

— Parallel axis theorem

10.6 Torque



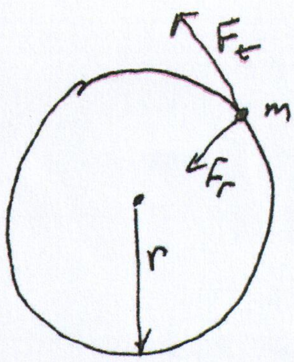
torque $\equiv \tau$

$= rF\sin\phi = Fd$

$\sum\tau = \tau_1 + \tau_2 = F_1 d_1 + F_2 d_2$

— for more than one torques.

10.7 Relationship between torque and angular acceleration



$F_t = m a_t$

$\tau = F_t r = m a_t r$

But $a_t = r\alpha$

$\therefore \tau = m(r\alpha)r$

$= mr^2\alpha$

$$\boxed{\tau = I\alpha}$$

$\therefore \tau \propto \alpha$

Torque is proportional to its angular acceleration, and the proportionality constant is the moment of inertia

Extend it to rigid object of arbitrary shape

$dF_t = dm\, a_t$

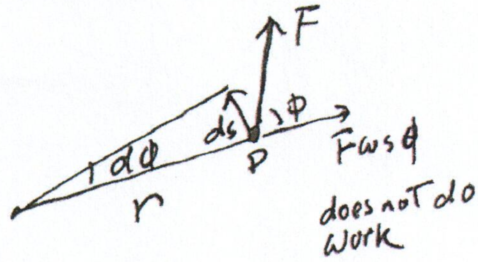
$d\tau = r\,dF_t = a_t r\,dm = r\alpha r\,dm = \alpha r^2 dm$

$\sum\tau = \int d\tau = \int \alpha r^2 dm = \alpha \int r^2 dm$

$\therefore \sum\tau = I\alpha$

10.8 . Work, Power and Energy in rotational motions

$$dW = F \cdot ds = \underbrace{F \sin\phi\, r\, d\theta}_{\tau\, d\theta}$$



$$\therefore\ dW = \tau\, d\theta$$

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} \quad \therefore \quad P = \frac{dW}{dt} = \tau \omega$$

$$\sum \tau = I\alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\theta}\frac{d\theta}{dt} = I \frac{d\omega}{d\theta}\omega$$

$$\sum \tau\, d\theta = dW = I\omega\, d\omega$$

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

Work - Kinetic energy theorem for rotational motions

10.9 Rolling motion of a rigid object

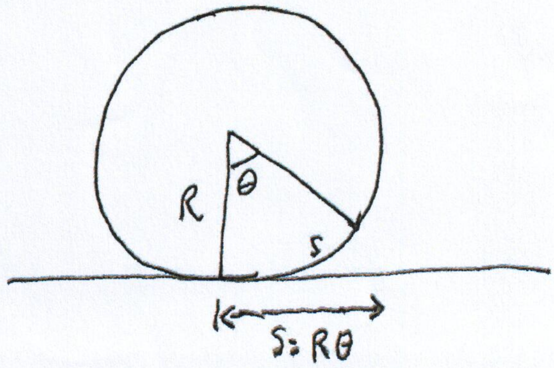
Fig 10-27, P.317

① the center of mass moves

$S = R\theta$

② the linear speed of CM is

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$



for a cylinder or sphere rolls without slipping

→ Condition for pure rolling

$$a_{cm} = \frac{dV_{cm}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

Check also Fig 10.28 (p 317)

The total Kinetic energy of the rolling cylinder:

$$K = \frac{1}{2} I_p \omega^2$$

$I_p \equiv$ moment of inertia about point P.

$$= \frac{1}{2}(I_{cm} + MR^2)\omega^2$$

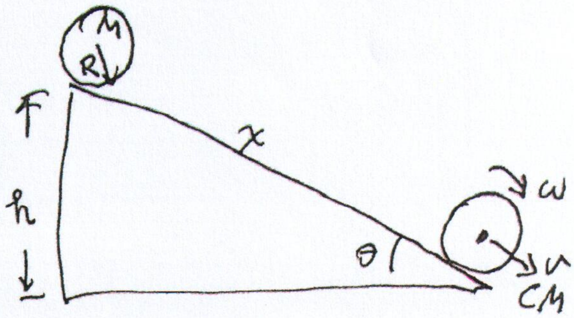
$$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \underbrace{\frac{1}{2} I_{cm}\omega^2}_{\text{Rotation about center of Mass}} + \underbrace{\frac{1}{2} M V_{cm}^2}_{\text{translation of center of mass}}$$

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M V_{cm}^2$$, But $V_{cm} = R\omega$

$$= \frac{1}{2} I_{cm}\left(\frac{V_{cm}^2}{R^2}\right) + \frac{1}{2} M V_{cm}^2$$

$$= \frac{1}{2}\left(\frac{I_{cm}}{R^2} + M\right) V_{cm}^2$$



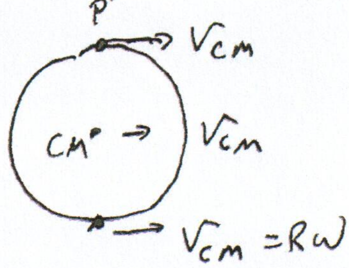
$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}\left(\frac{I_{cm}}{R^2} + M\right)V_{cm}^2 + 0 = 0 + Mgh$$

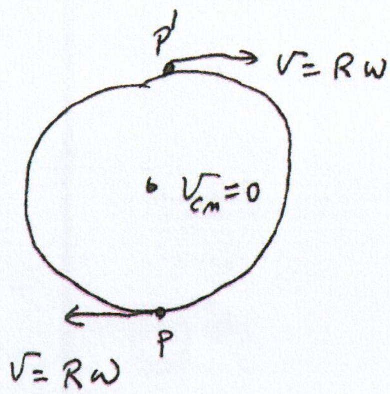
$$V_{cm} = \left(\frac{2gh}{1 + \frac{I_{cm}}{MR^2}}\right)^{\frac{1}{2}}$$

Page 318, Fig 10.29

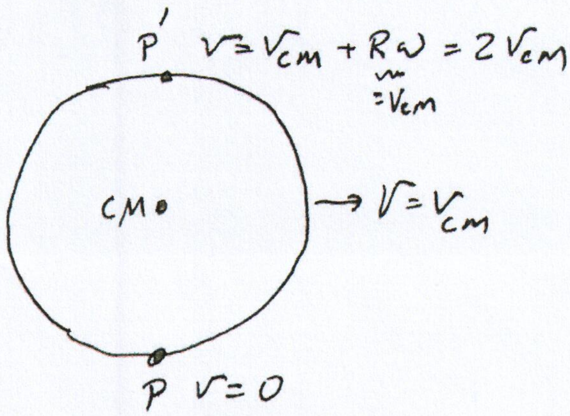
(a) pure translation



(b) Pure rotation



(c) Combination of translation + rotation



a combination of (a) + (b)