Inventory Analysis and Management

Single-Period Stochastic Models

6.1. Newsvendor Models: Optimality of the Base-Stock Policy

A newsvendor problem is a single-period stochastic inventory problem. First assume that there is no initial inventory, i.e., \( I_0 = 0 \). Let

- \( c \) be the cost to buy each unit;
- \( h \) be the holding cost for each unit of inventory left over at the end of the period;
- \( \pi \) be the shortage cost for each unit of unsatisfied demand in the period;
- \( D \) be the demand of the period such that \( D \) is a continuous random variable \( \sim F \);
- \( \nu(q) \) be the expected cost of the period for order quantity \( q \).

The objective is to minimize the total cost of the period, i.e.,

\[
\min \nu(q) = cq + hE[q-D] + \pi E[D-q].
\]

As shown in the second set of notes, \( \nu(q) \) is a differentiable\(^\dagger\) convex function in \( q \). The first-order necessary condition, which is also sufficient, gives

\[
c + hF(q) - \pi F^c(q) = 0,
\]

i.e.,

\[
F(q^*) = \frac{\pi - c}{\pi + h}.
\]

The minimum cost is achieved at ordering \( q^* \) that satisfies (3).

Suppose that \( I_0 \geq q^* \). If we order any quantity \( y \), the expected total cost per period = \( \nu(I_0+y) + cy \geq \nu(q^*) \). Therefore, it is optimal not to order if \( I_0 \geq q^* \). Now consider \( 0 < I_0 < q^* \). Let us compare the two policies of ordering \( q^*-I_0 \) and ordering \( y \neq q^*-I_0 \). The expected total cost of ordering \( q^*-I_0 = \nu(q^*) - cq^* + c(q^*-I_0) = \nu(q^*) - cI_0 \), while the expected total cost ordering \( y = \nu(I_0+y) - c(I_0+y) + cy = \nu(I_0+y) - cI_0 \geq \nu(q^*) - cI_0 \). It is optimal to order \( q^*-I_0 \). The result shows that the optimal policy in (3) is of base-stock type: If \( I_0 \geq q^* \), order nothing; otherwise, order \( q^*-I_0 \).

Example 6.1.1. Consider the standard Newsvendor problem with \( c = $1/\text{unit} \), \( h = $3/\text{unit}, \)

\[^\dagger\] \( \frac{dE[q-D]}{dq} = \nu(q); \quad \frac{dE[D-q]}{dq} = -F^c(q), \) where \( F^c(q) = 1 - F(q). \)
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and $\pi = \$2/\text{unit}$, and $D \sim \text{uniform}[0, 100]$. Find the optimal order quantity.

**Sol.** First of all, it is intuitively clear that the optimal order quantity $q^*$ is within $[0, 100]$. Then there are different ways to compute $E[q-D]^+$ and $E[D-q]^+$. Let $f$ be the density function of $D$. A standard way based on the form of $[q-D]^+$ and the definition of density function is that

$$E[q-D]^+ = \int_0^{100} [q-x]^+ f(x)dx = \int_0^q (q-x)f(x)dx + \int_q^{100} [q-x]^+ f(x)dx = \frac{q^2}{200}. $$

Another way based on conditional probability is that

$$E[q-D]^+ = P(q > D)E[(q-D)^+|q > D] = \left(\frac{q}{100}\right)\left(\frac{q}{2}\right) = \frac{q^2}{200}. $$

One way or another, $E[D-q]^+ = P(D > q)E[(D-q)^+|D > q] = \left(\frac{100-q}{100}\right)\left(\frac{100-q}{2}\right) = \frac{(100-q)^2}{200}.$

Therefore, $v(q) = q + \frac{3q^2}{200} + \frac{2(100-q)^2}{200}$. The first-order condition is that

$$v'(q) = 1 + \frac{3q}{100} - \frac{2(100-q)}{100} = 0 \quad \Rightarrow \quad \frac{3q+2q}{100} = 2 - 1, $$

i.e., $q^* = \frac{100}{5} = 20.$

Consider a variant of the minimum cost problem. Let $D$ be the demand of a particular newspaper at a news stand such that $D \sim F$ and $E(D) = \mu$. Each piece of newspaper is bought at $c$ and sold at $p$. There is no salvage value of a piece of newspaper at the end of a day. Let $r(q)$ be the expected profit if the newsstand orders $q$ pieces. Find $q^*$ that maximizes $r(q)$.

$$\max r(q) = p E[\min(D, q)] - cq. \quad (4)$$

We are going to express (4) in a form similar to (1): First ignore the order quantity. Suppose every unit of demand is satisfied. Then the expected profit is $(p-c)E(D) = (p-c)\mu$. This is an over estimate. Some demands are unsatisfied when $q$ is smaller than $D$; some items are bought but are left over as scrap when $q$ is larger than $D$. The expected profit loss in the former case is $(p-c)E(D-q)^+$ and the expected profit loss in the latter case is $cE(d-D)^+$. Thus,

$$\max r(q) = (p-c)\mu - [(p-c)E(D-q)^+ + cE(q-D)^+]. \quad (5)$$

The derivation can be more analytical. Note that
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\[
\min (D, q) = D + q - \max(D, q) = D - [D-q]^{+}, \quad \text{and} \\
q = D + [q-D]^{+} - [D-q]^{+}. \quad (6)
\]

\[
r(q) = pE[\min(D, q)] - cq = pE[D] - pE[D-q]^{+} - cE[D] - cE[q-D]^{+} + cE[D-q]^{+} \\
= (p-c)\mu - [(p-c)E[D-q]^{+} + cE[q-D]^{+}].
\]

As the first term is constant, to maximize \( r(q) \) is the same as minimizing

\[
v(q) = (p-c)E[D-q]^{+} + cE[q-D]^{+},
\]

which is in the same form of (1) without the fixed ordering cost. From (3), the optimal order quantity is given by

\[
F(q^*) = \frac{p-c}{p}. \quad (8)
\]

Remark 6.1.1. There are equivalent forms of \( r(p) \) other than the one given above. Instead of (6),

\[
\min (D, q) = D + q - \max(D, q) = q - \max(0, q-D) = q - [q-D]^{+}. \\
r(q) = pE[\min(D, q)] - cq = pq - p[q-D]^{+} - cq,
\]

which leads to the same result as (8).

Example 6.1.2. Consider a similar context as Example 6.1.1, except that each unit of item is sold for $5, and the objective is to maximize the expected cost.

Sol. We are going to use the equivalent expression of \( r(q) \) shown in Remark 6.1.1.

\[
r(q) = pE[\min(D, q)] - cq - hE[q-D]^{+} - \pi E[D-q]^{+} = (p-c)q - (p+h)E[q-D]^{+} - \pi E[D-q]^{+}.
\]

With \( E[q-D]^{+} \) and \( \pi E[D-q]^{+} \) found in Example 6.1.1,

\[
r(q) = (p-c)q - (p+h)\frac{q^{2}}{200} - \pi \frac{(100-q)^{2}}{200}, \quad \text{and} \quad r'(q) = (p-c) - (p+h)\frac{q}{100} + \pi \frac{(100-q)}{100}.
\]
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The first-order condition \( r'(p) = 0 \) gives \((p-c) + \pi = \frac{q}{100} (p+h+\pi) \); i.e., \( \frac{q}{100} = \frac{p+\pi-c}{p+\pi+h} \). 

\[
\frac{5+2-1}{5+2+3} = 0.6, \text{ which gives } q^* = 60.
\]

6.2. Fixed-Cost Models: Optimality of the \((s, S)\) Policy

Consider the same cost structure of \( c, h, \pi, D, \) and \( v(q) \) leading to (1), plus the additional setup cost \( K \) for placing an order (i.e., \( q > 0 \)). The objective is still to minimize the total cost in the period. In this case, what should the optimal inventory policy be?

In the newsvendor model without any setup cost, the base-stock policy is always to order up to \( q^* \) if \( I_0 < q^* \), and order nothing otherwise. However, with the setup cost, the optimal ordering policy can be different. Let \( S^* \) be the optimal on-hand stock in this case, no matter how \( S^* \) is found. If \( I_0 \) is only “slightly” less than \( S^* \), i.e., the benefit of having on-hand \( S^* \) over \( I_0 \) is not too big, it may not be worthwhile to pay the setup cost \( K \) (and additional variable cost on ordering) to change the on-hand inventory to \( S^* \). The following analysis expresses the above idea rigorous and derivates the optimal inventory policy.

First let \( v(q) = cq + hE[q-D]^+ + \pi E[D-q]^+ \) be the sum of the variable cost in ordering, the expected inventory holding cost, and the expected shortage cost. Recall that \( v(q) \) is a convex function with a well-defined minimum. Let \( S^* \) be the value of \( q \) that minimizes \( v(q) \). Let \( s^* < S^* \) such that \( v(q^*) = K + v(S^*) \). Refer to Figure 1 for the definitions of \( s^* \) and \( S^* \). We claim that the optimal inventory (production) policy is of \((s, S)\) type, i.e.,

- order \( S^*-I_0 \), if \( I_0 < s^* \),
- order nothing, if \( I_0 \geq s^* \).

\[\text{Figure 1. The Definition of } s^* \text{ and } S^*\]
To establish the claim, consider four cases: (i) \( I_0 < s^* \), (ii) \( I_0 = s^* \), (iii) \( s^* < I_0 \leq S^* \), and (iv) \( I_0 > S^* \).

**Case (i) \( I_0 < s^* \)**: Ordering incurs the cost \( K + c(S^* - I_0) \) for the order. The on-hand inventory becomes \( S^* \), and the expected inventory holding and shortage costs will be \( v(S^*) - cS^* \). The deduction of \( cS^* \) is necessary because \( v(S^*) \) includes the variable cost to buy \( S^* \) items, which is fact is extra. Thus, the expected total cost of ordering is \( K + c(S^* - I_0) + v(S^*) - cS^* = K + v(S^*) - cI_0 \). By a similar argument, the expected total cost of not ordering is \( v(I_0) - cI_0 \). Since \( I_0 < s^* \), \( v(I_0) > v(s^*) = K + v(S^*) \). The \((s, S)\) policy is optimal in this case.

**Case (ii) \( I_0 = s^* \)**: From the argument, ordering or not makes no difference. The \((s, S)\) policy is optimal in this case.

**Case (iii) \( s^* < I_0 \leq S^* \)**: The two expressions of the expected total cost remain valid, i.e., the expected total cost for is \( K + v(S^*) - cI_0 = v(s^*) - cI_0 \) and the expected total cost for not ordering is \( v(I_0) - cI_0 \). For \( I_0 \) in the given range, \( v(I_0) < v(s^*) \). It is better not to order, and the \((s, S)\) policy is optimal in this case.

**Case (iv) \( I_0 > S^* \)**: Clearly there is no point to order and the \((s, S)\) policy is optimal.

**Example 6.2.1.** Consider the same context as Example 6.1.1. Now there is a fixed cost \( K = $5 \) per order. Find the order quantity \( q^* \) that minimizes the expected cost.

**Sol.** \( v(q) = cq + hE[q - D]^+ + \pi E[D - q]^+ \). In the context of the problem,

\[
v(q) = cq + h \frac{q^2}{200} + \pi \left( \frac{100 - q^2}{200} \right) = q^2 - 40q + 4000.
\]

By construction, \( v(S^*) = 0 \), i.e., \( S^* = 20 \), and \( v(S^*) = \frac{(20)^2 - (40)(20) + 4000}{40} = 90 \). \( s^* \) is defined by the value of \( q < S^* \) such that \( v(s^*) = K + v(S^*) = 95 \).

\[
\left( s^* \right)^2 - 40s^* + 4000 = 95, \quad i.e., \quad s^* = 20 - \sqrt{200} = 5.8579.
\]

The optimal policy is to order \( 20 - I_0 \) if \( I_0 < 5.8579 \), and do nothing otherwise.