Chapter 7. Electrodynamics.

$\sigma$: conductivity?

$\Rightarrow$ Microscopic view of Ohm's law.

The current density can be expressed in terms of the free electron density as $J = neV_d$

$I = \frac{E}{d}$, $I/A = J$

$n = \frac{N \text{ (atoms/mole)} \cdot P \text{ (kg/m}^3\text{)}}{A \text{ (kg/mole)}}$

1. $F = ma = \frac{mV_d}{d} = \frac{mV_d}{V_F}$

2. $F = qE$

3. Kinetic energy, $E_k = \frac{1}{2}mV_F^2$.

Then the current density

$J = neV_d = \sigma E$

$F = ma = qE = \frac{mV_d}{d}V_F$

$\sigma = \frac{neV_d}{E} = ne\frac{EEd}{mV_F} = \frac{ne^2d}{mV_F} = \sigma_c$

$\sigma$: conductivity.
§ 7.1.1 Force on charge will fast/slow their velocity, so the current density $J$ is proportional to the force per unit charge ($f$), $f = \frac{F}{q}$.

$$J \propto f = \sigma f, \quad P = \frac{1}{\sigma} : \text{resistivity}.$$

if we consider the electrodynamics of electrical/magnetic force, the current density.

$$J = \sigma f = \sigma \frac{F}{q} = \sigma (\vec{E} + \vec{v} \times \vec{B})$$

Ohm's Law $J = \sigma E$.

§ 7.1.2 Electromotive Force. ($\text{Emf}$)

It is the work per unit charge done by force/on charge.

Dimensions: energy per charge.

1. The emf of a battery

$$E = V = \int_{a}^{b} E \cdot dl,$$

$$qE(b-a)/q = \int E \cdot dl$$
(2) \( E = \frac{1}{\mu} \oint_a b \ \vec{E} \cdot d\vec{l} \) (Electromotive force)

\[ = \frac{1}{\mu} \oint b(\vec{v} \times \vec{B}) \cdot d\vec{l} \]

Velocity.

Ex 7.1 A cylindrical resistor.

\[ I = JA = \sigma EA \quad \Rightarrow \quad I = \sigma A \frac{V}{L} \]

Ex 7.2 Two long cylinders (radius a & b).

Conductivity \( \sigma \) maintained at

a potential difference \( V \).

§ The \( E \)-field between \( \frac{9}{6} \) radius.

1. \( E = \frac{\lambda}{2\pi \varepsilon_0 \delta} \) (P.2.35)

2. The current is therefore

\[ I = \oint \vec{J} \cdot d\vec{a} = \oint \sigma c E \, d\vec{a} = \frac{\sigma c A}{2\pi \varepsilon_0} \oint \frac{1}{\delta} d\vec{a} \]

\[ d\vec{a} = 2\pi \delta L \]
The total current
\[ I = \frac{\sigma E}{\varepsilon_0} L \]
if the potential between \( b/a \)
\[ V = -\int E \cdot dl, \quad -\Delta \phi = \varepsilon \]
\[ = -\int_{a}^{b} \frac{\varepsilon}{2\pi \varepsilon_0} \cdot dl = \frac{-\varepsilon}{2\pi \varepsilon_0} \ln \frac{b}{a} \]

Then the resistivity \( V = IR \)
\[ R = \frac{\ln b/a}{2\pi \varepsilon_0 L}, \quad \sigma = \frac{\ln b/a}{2\pi RL} = \frac{1}{\rho} \]

\[ E_{\text{m}} = \frac{\Delta \Phi}{\Delta t} L \]

We can discuss magnetic flux \((B \cdot da)\)
in the term at for the charge it sweeps
an area \( V \cdot \Delta t \cdot L \) in time \( \Delta t \).

* The charge in magnetic flux associated with
this motion is \( \Delta \Phi = BA = B \frac{\Delta \Phi}{\Delta t} \Delta t + L \)

* The rate of change of magnetic flux.
\[ \frac{dE}{dt} = \frac{\frac{d\Phi}{dt}}{L} = \frac{\Delta \Phi}{\Delta t} \]

\( E_{\text{m}} \) is the magnetic flux per time.
(\text{Lenz's law})
3.7.1 Microscope. Ohm's Law.

\[ E = \frac{d\Phi}{dt} \text{ (E-field)} \]

1. \( E \)-field \( \rightarrow \) \( J = \sigma E = \sigma \dot{f} \), force per unit charge. \( \overline{E} = E A \)

2. \( B \)-field \( \rightarrow \) \( E = -\frac{d\Phi}{dt} \), Electromotive force, work per unit charge. \( \overline{E}_B = BA \)

* Faraday's Law

3. Lenz's Law: We have not specified the direction of the emf, which is provided by Lenz's Law

\[ |\mathcal{E}| = \frac{d\Phi}{dt}. \]

* The induced emf acts in such way as to oppose the change of time in flux, generate currents in circuit.

\[ |\mathcal{E}| = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int B \cdot dA \quad \text{Faraday's Law.} \]

of electromagnetic induction.

\[ F_i = L \frac{d\Phi}{dt} \leftrightarrow L \leftrightarrow \text{mass, } m \frac{dx}{dt^2} = F_i \]

4.2.1 Faraday's Law

"A changing magnetic field induce an electric field"

\[ \frac{d\Phi_B}{dt} \rightarrow \frac{d\mathcal{E}}{dt} ? \]

The emf is equal to the rate of change of the flux.
\[ E = \frac{1}{\mu} \int \mathbf{E} \cdot d\mathbf{l}, \quad \mathbf{E} = \frac{\mathbf{F}}{q} = \frac{\mathbf{q}}{q} \]

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\mathbf{B}}{dt} \]

Then \( E \) is related to the charge in \( B \) by the equation:

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \]

We can apply to the Stokes' theorem:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Note:

\[ \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{\mu} \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]

The circuit can be through of any closed geometric path in space, not necessarily coincident with an electric field.

So \[ \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]

Then \[ \int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]
The induced electric field.

Faraday's discovery tells us two kinds of electric fields.

1) electric charges (e)

2) changing magnetic field \( \frac{dB}{dt} \), (L) mass

The latter can be found by Faraday and Ampere's law.

\[
\begin{align*}
\nabla \times E & = 0 \\
\n\nabla \times B & = \mu_0 J \\
\n& = -\frac{\partial B}{\partial t} \\
\n& = 0
\end{align*}
\]

5/14

Chapter 7. Ohm's Law

Faraday's Law

1) The E-field can be originated from two sources.

A. Charges. \( q \)

B. Changing magnetic field \( \frac{dB}{dt} \) current changes with time.

\[
\begin{align*}
\int \mathbf{E} \cdot d\mathbf{l} & = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \\
& = -\frac{d}{dt} \mathcal{E} = -A \frac{dB}{dt} \\
& = -\int \frac{\partial B}{\partial t} dA
\end{align*}
\]
Pro. 7.12 A long solenoid of radius $a$, is driven by an alternating current, so that the field inside is $B(t) = B_0 \cos(\omega t) \hat{z}$. A circular loop of wire, radius $\frac{a}{2}$, resistance $R$, is placed inside the solenoid. Find the current induced in the loop.

1. If the magnetic flux is $\Phi = BA$, $A$: observed area $= \pi a^2/4$

2. Then the emf is

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left[ \frac{\pi a^2}{4} B_0 \cos(\omega t) \right]
\hspace{1cm}
\begin{align*}
\varepsilon &= \frac{\pi a^2}{4} B_0 \sin(\omega t)
\end{align*}
$$

3. $\varepsilon = IR$, $I = \frac{\pi a^2}{4R} B_0 \sin(\omega t)$

P.7.13 A square loop of wire, with sides of length $a$, lies in the first quadrant of the $xy$ plane. In the region is a non-uniform time-dependent magnetic field $B(y, t) = ky^3 t^2 \hat{z}$. Find emf?

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left( B y^3 \cdot dA \right) = -\frac{d}{dt} \int_{y^3} \frac{1}{4} y^4. \frac{1}{4} y^4 \cdot dA, \hspace{1cm} dA = dy \cdot dz$$

$$\varepsilon = -\frac{d}{dt} \left. \frac{1}{4} y^4 \right|_{y, z = a} = -\frac{kta^5}{2}$$
\[ F_E = qE. \]

\[ W = \int F_E \cdot dl \Rightarrow \frac{W}{q} = \int E \cdot dl = \int \vec{E} \cdot d\vec{l} \]

**Ex 7.7** If the B-field is changing with time, what's the induced E-field?

From Faraday's law of B\(\times\)E:

\[
\phi \frac{d\vec{E}}{dt} = -A \frac{dB}{dt}
\]

clockwise

\[ E \cdot 2\pi s = -s a^2 \frac{dB}{dt} \quad \text{Therefore} \quad \vec{E} = -\frac{s}{2} \frac{dB}{dt} \]

**Ex 7.8** A line charge \( \lambda \)

1. \[ \phi \vec{E} \cdot d\vec{l} = -\pi a^2 \frac{dB}{dt} \]

2. The torque is \( \vec{T} \times \vec{F} = b \times qE \)

\[ = b \times (2\pi \lambda) E = N \]

Then the torque is

\[ N = \phi b \lambda \vec{E} \cdot d\vec{l} = b \lambda \pi a^2 \frac{dB}{dt} = \pi b \lambda (-\pi a^2) \frac{dB}{dt} \]

\[ = -\pi a^2 b \frac{dB}{dt} \]
3. The total angular momentum is

\[ \sum N \frac{dx}{dt} = -\lambda \pi a^2 b \int_{B_0}^{B} dB = \lambda \pi a^2 b B_0 \]

Ex 7.9: An infinity long straight wire carries a slowly varying current \( I(t) \). Determine the induced electric field as a function of the distance \( s \), from the wire.

From Ampere's Law, \( B(t) = \frac{\mu_0 I(t)}{2\pi s} \).

I. the induced E-field

\[ \frac{d\mathbf{E} \cdot d\mathbf{l}}{dt} = -\frac{d\mathbf{B}}{dt} = -\frac{d}{dt} \int_{s_0}^{s} B(t) \, dA = -\frac{d}{dt} \int_{s_0}^{s} \frac{\mu_0 I}{2\pi s'} \, ds' \, dl \]

\[ = -\frac{d}{dt} \left( \frac{\mu_0 I}{2\pi} \right) \ln s' \]

\[ \int_{s_0}^{s} \frac{1}{s} \, ds = \ln s \]

\[ \Rightarrow \left[ \mathbf{E} \cdot d\mathbf{l} \right] = -\frac{d}{dt} \left( \frac{\mu_0 I}{2\pi} \right) \left[ \ln s - \ln s_0 \right] \]

\[ E = \frac{\mu_0}{2\pi} \left( \ln s - \ln s_0 \right) \frac{dI}{dt} \]
Example review

1. E-field $\propto \frac{dI}{dt}$

2. E-field $\propto \frac{-dB}{dt}$

3. $\text{Emf} = -\frac{d\Phi}{dt}$

1.2.3 Inductance

1. For a given electrical circuit, the B-field produced by any point is proportional to the current flowing in the circuit, changing with time.

2. The magnetic flux $\Phi$ linking any closed path is proportional to $I$, We may write as $\Phi = LI = BA$.

3. If the magnetic field, the flux varies with time, if $\Phi(t, I)$ function.

   \[ \frac{d\Phi(t, I)}{dt} = \frac{d\Phi}{dI} \frac{dI}{dt} \equiv L \frac{dI}{dt} \left( E - \frac{dE}{dt} \right) \text{ for magnetic field } B(t). \]

\[ \equiv L \frac{d^2\Phi}{dt^2} \text{ for charges } \nabla \cdot E = \frac{\rho_s}{\varepsilon_0}. \]

\[ \Rightarrow F = ma = m \frac{d^2x}{dt^2}. \]

4. A emf $E$ is also induced in the circuit as current $I$.

   \[ E = -\frac{d\Phi}{dt} = -L \frac{dI}{dt}. \]
§ 7.2.3 Inductance

1. Magnetic flux $\Phi = LI(t)$

2. Flux varies with time

$$\varepsilon = \frac{d\Phi}{dt} = -L \frac{dI}{dt}$$

3. The negative sign reminds us that this induced emf tends to oppose the change of current.

4. The unit of $L$ is called Henry

$$1 \text{ henry} = 1 \frac{\text{volt-second}}{\text{ampere}} = 1 \frac{\text{voltage} \cdot \text{second}}{\text{ampere}}$$

Symbol $L$.

1. If $I$ is the current flowing at any time $t$ after the switch $k_1$ is closed, we have $\varepsilon_0 = L \frac{dI}{dt} + RI$ or $L \frac{dI}{dt} + RI = \varepsilon$, solving the differential eq, with the initial condition:

$$\begin{cases} 
  t = 0, \quad I = 0 \\
  I = I_0 \left( 1 - \exp \left( \frac{-Rt}{L} \right) \right) = I_0 - I_0 \exp \left( \frac{-Rt}{L} \right)
\end{cases}$$

2. where $I_0 = \frac{\varepsilon}{R}$ as $t \to \infty$. The time $\frac{L}{R}$ is the time constant or relaxation time $\tau = \frac{L}{R}$.
Then the eq can be expressed as

\[ I = I_0 - I_0 \exp^{-\frac{t}{2}} \]

* When connected to \( K_2 \), we have

\[ L \frac{di}{dt} + RI = 0 \implies I = I_0 \exp^{-\frac{Rt}{L}} \]

where the initial condition is \( I = I_0 \), at \( t = 0 \).

Ex. Calculate the inductance \( L \) of a long solenoid. Magnetic field inside a long solenoid is

\[ B = \frac{\mu_0 IN}{S} = \frac{N \mu_0 I}{4\pi S} \approx \frac{N \mu_0 I}{S}, \quad S: \text{length} \]

\[ \Phi = BA = \frac{\mu_0 NI}{S} \pi a^2 \]

\[ L = \frac{NI}{I} = \frac{\mu_0 \pi N^2 a^2}{S} = \mu_0 \pi a^2 n \cdot N. \]

§ 7.2.3 Mutual inductance & Neumann's Formula.

(互感)

If we consider more than one circuit, we generalized the equation to

\[ \frac{d\Phi_k}{dt} = \frac{d\Phi_l}{dt} = M_{kl} \frac{di_k}{dt} \]

where \( M_{kl} \) is the mutual inductance between circuit \( k \) & circuit \( l \).
The unit of Ω is heny, \( M_{xx} = \frac{d\phi_{zz}}{d\alpha} = L_{xx} \)

\[ \phi_{zz} = M_{zz} I_z, \quad M_{zz} = \frac{\phi_{zz}}{I_z} \]

The induced emf in circuit 2, \( E = -M_{zz} \frac{dI_z}{dt} \)

To drive Biot-Savart + Neumann's formula, we can find

\[ E = - \oint \frac{2\mathbf{B}}{\alpha} \cdot d\mathbf{A} = -\int \mathbf{x} \cdot \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{A} \]

Stoke's theorem

\[ \oint \mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{A}, \quad \text{where} \quad \mathbf{A} = \frac{\mathbf{m}_0}{{\mu}_0} \int \frac{I_z \cdot ds}{r} \]

\[ E = - \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{A} \]

\[ = - \int_{\alpha}^{\beta} \frac{\mathbf{m}_0}{\mu_0} \int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{A} \]
1.2 Mutual inductance

\( I_1(t) \) circuit 1
\( I_2(t) \) circuit 2
\( \mathbf{B}_s = \nabla \times \mathbf{A}_s \)

* The flux is induced by circuit 1.

\[ \mathbf{E} = \frac{1}{\mu_0} \oint \mathbf{A}_s \cdot d\mathbf{A} = -\frac{1}{\mu_0} \oint \frac{d\mathbf{A}}{dt} \cdot d\mathbf{A} \]

* To derive Neumann's formula

Then the B-field can be replaced as

* The vector potential

\[ \mathbf{A}_s = \frac{\mu_0}{4\pi} \oint \frac{I_1 \cdot ds}{r} \]

So the Emf:

\[ \mathbf{E} = -\frac{d}{dt} \oint \frac{\mathbf{B}_s}{\mu_0} \cdot \frac{I_1}{4\pi} d\mathbf{A} \]

\[ M_{21} = \oint \oint \oint \frac{d\mathbf{s} \cdot d\mathbf{A}}{r} \]

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We could get the definition of $M$ is $M = N_1 \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{s} \cdot d\mathbf{r}}{r}$, where $d\mathbf{s}$ & $d\mathbf{r}$ are two elements of length & $r$ is the distance between them.

A. $M_{1i}$ is purely geometrical quantity, having to do with the sizes, shapes, and relative position.

B. The mutual inductance is unchanged if we switch the roles of loop $A$ & $S$.

Then $M_{21} = M_{12}$

$$\sum M_{ij} \Rightarrow \begin{bmatrix} M_{11} + M_{12} + M_{13} \\ M_{21} + M_{22} + M_{23} \end{bmatrix} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$$

§ 7.2.4 Magnetic energy in terms of circuit parameter.

- We now apply it to $n$ coupled circuits, then the flux changes are directed related with changes in the currents in the $n$ circuits.

$$d\Phi_j = \sum \frac{d\Phi_{ij}}{di_j} \, di_j = \oint M_{ij} \, di_j$$

\[ \text{Diagram:} \]

- Circuit diagram showing coupled inductors with a common magnetic flux $\Phi_x$.
* for stationary circuit, no mechanical work \((\text{dw} = 0)\) is associated with the flux changes \(d\Phi\).

Then \(\text{dw}\) is equal to \(dU\) (the change in magnetic flux).

* If all currents are built up at all the same traction \((\alpha)\) of their final values.

\[
I_i' = \alpha I_i, \quad d\Phi = \Phi_i' d\alpha
\]

\[
\int d\text{w} = \int_0^1 d\alpha \sum I_i' \Phi_i' = \frac{1}{2} \sum_{i=1}^n I_i \Phi_i
\]

Thus the magnetic energy of the system is

\[
U = \frac{1}{2} \sum I_i \Phi_i
\]

\[
= \frac{1}{2} \sum_{i,j} M_{ij} I_i I_j
\]

\[
= \frac{1}{2} M I^2, \quad \frac{1}{2} LI^2
\]

\[\text{Note that for linear media}\]
* Mutual Inductance for linear media.

\[ M_{ij} = M_{ji} & M_{ii} = L_i, \quad N = \frac{1}{2} MI^2 \]

* Magnetic energy in terms of field vector.

We assume a single loop, then the flux \( \Phi \) may be expressed as

\[ \Phi = \oint B \cdot \hat{n} \, ds = \oint \mathbf{A} \cdot \hat{n} \, ds = \oint \mathbf{A}_s \cdot \, dl \]

\[ = \oint \mathbf{A} \cdot \, dl \] (for single)

where, \( c \) is the enclose path

\[ U = \frac{1}{2} \sum I_i \Phi_i \]

\[ = \frac{1}{2} \sum A \mathbf{A} \cdot dl \]

Note: We can change \( I_i \, dl \) with,

1. \( j \cdot da \cdot dl = j \, dv \) \( \left( J = \frac{i}{A}, \; j \, da = I \right) \)

2. \( \sum \mathbf{A} \rightarrow \int \mathbf{A} \rightarrow \int \mathbf{A} \, dv \)

\[ \text{sum} \rightarrow \text{fraction} \rightarrow \text{Integration} \]

\[ \Rightarrow U = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{A} \, dv \]

Apply \( \mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)
\[
\begin{align*}
\n\n\text{if } \nabla \cdot (\vec{A} \times \vec{B}) \\
= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \\
= \vec{B} \cdot \vec{B} - \vec{A} \cdot (\vec{m}_0 \vec{J}) \\
\vec{m}_0 \vec{A} \cdot \vec{J} = \vec{B} \cdot \vec{B} - \nabla \times (\vec{A} \times \vec{B})
\end{align*}
\]

We obtain

\[
U = \frac{1}{2 \mu_0} \int_V \vec{B} \cdot \vec{B} \, dV - \frac{1}{2 \mu_0} \int_A \nabla \cdot (\vec{A} \times \vec{B}) \, dV
\]

\[
= \frac{1}{2 \mu_0} \int_V \vec{B} \cdot \vec{B} \, dV.
\]

\[
U_v = \frac{U}{V} = \frac{1}{2 \mu_0} |\vec{B}|^2
\]

\[
\int \nabla \times (\vec{A} \times \vec{B}) \, dV = \oint_S \vec{A} \times \vec{B} \cdot \vec{n} \, ds
\]

when \( S \) is the surface bounds the volume.

\[
\vec{m}_0 = \frac{U}{A} \rightarrow 0 \text{ (太小了,量不到)}
\]

We can take any region larger than this for current density \( \vec{J} \) is zero out there.
because \( \mathbf{B} \) falls off at least as fast as \( \frac{1}{r^2} \)

\[ \mathbf{A} \sim \frac{1}{r} \]

\( \mathbf{A} \) \( \sim \) \( \frac{1}{r} \)

\( \text{Surface, } r^2 \)

Then the second term of surface integral vanishes

\[
W_{\text{mag}} = U = \frac{1}{2 \mu_0} \int \mathbf{B} \cdot \mathbf{B} \, d\mathbf{v} = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{H} \, d\mathbf{v}
\]

\( \Rightarrow \) \( U = \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \) for the case of isotropic

\[
U = \begin{bmatrix} u_{11} & \cdots & u_{13} \\ \\ \vdots & \ddots & \vdots \\ u_{31} & \cdots & u_{33} \end{bmatrix}
\]

tensor for the anisotropic

\[
W_{\text{ele}} = \frac{\varepsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, d\mathbf{v} = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{D} \, d\mathbf{v}
\]

**Ex.** A long coaxial cable carries current \( I \). Important

the outer current \( I \) \leftarrow \[ \text{inner current } I \rightarrow \]

Find the magnetic energy stored in a section of length \( L \).
According to Ampere's Law, the field between the cylinders:

\[ B = \frac{\mu_0 I}{2\pi p} \]

The energy per unit volume is

\[ U = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0} \left( \frac{\mu_0 I}{2\pi p} \right)^2 = \frac{\mu_0 I^2}{8\pi^2 p^2} \]

Total \( U = \int u \, dv = \int \frac{\mu_0 I^2}{8\pi^2 p^2} \cdot 2\pi p \, dp \)

\[ = \frac{\mu_0 I^2 l}{4\pi} \int \frac{1}{p} \, dp = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{b}{a} \]

The unit volume of cylinder is

\((2\pi p \, dp)l\)

* Find the inductance \( U = \frac{1}{2} LI^2 \)

\[ L = \frac{2U}{I^2} = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a} \]
1.3 Maxwell Equation

1.3.1 Electro dynamics before Maxwell (old rules)

1. Gauss's Law \( \nabla \cdot E = \frac{\rho}{\varepsilon_0} \)

2. Gauss's Law \( \nabla \cdot B = 0 \)

3. Ampere's Law \( \nabla \times B = \frac{\mu_0}{c} \mathbf{J} - \frac{\partial \mathbf{E}}{\partial t} \)

4. Faraday's Law \( \nabla \times E = -\frac{\partial B}{\partial t} \) (changing B-field)

5. The divergence of curl is zero. \( \nabla \cdot (\nabla \times A) = 0 \)

These eqs. represent the state of EM theory over a century.

*From the eqs. of 5. the old rules that divergence of curl is always zero.

1. \( \nabla \cdot (\nabla \times E) = 0 \) Electric statics.

\[ \nabla \cdot (-\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0 \]

2. \( \nabla \cdot \mathbf{J} = 0 \), \( \nabla \cdot (\nabla \times A) = 0 \)

\[ \nabla \cdot (\nabla \times B) = 0 \]

\[ \nabla \cdot (\mu_0 \mathbf{J}) = \mu_0 (\nabla \cdot \mathbf{J}) = ? (0) \]
7.3.2 How Maxwell fixed Ampere's Law

Continuity eq. & Ampere's Law, they were all valid even in time varying situations.

\[ \nabla \cdot J + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{& realize Ampere's law was in consistent with the continuity eq. ?} \]

if we taken the \( \nabla \cdot (\nabla \times \mathbf{B}) \) on both sides

\[ \nabla \cdot (\nabla \times \mathbf{B}) = 0 \]

This is indeed true of \( \rho \) does not change with time.

* But it is not true when \( \rho \) is changing with time. Maxwell's suggested a way out of this.

Using Gauss's Law

\[ \nabla \cdot \mathbf{D} = \rho/\varepsilon_0 \]

or

\[ \nabla \cdot (\nabla \times \mathbf{E}) = 0 \iff \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]

P.175

\[ \nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{J}) = \nabla \cdot \mu_0 \left[ \mathbf{D} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right] = 0 \]
The term of $\frac{\partial D}{\partial t}$ has the dimensions of the current density & called "displacement current" $= \varepsilon_0 \frac{\partial E}{\partial t}$.

\[ \frac{\partial D}{\partial t} = \varepsilon_0 E + P \rightarrow \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} \]

Let consider a simple circuit such as parallel plates capacitor connected to a battery.

1. $I = \frac{dQ}{dt}$, $J = \frac{I}{A}$

\[ E = \frac{V}{d} = \frac{Q}{Ad} \]

\[ \frac{\partial E}{\partial t} (Ne) \Rightarrow \varepsilon_0 \frac{\partial E}{\partial t} \left( \frac{Q}{\varepsilon_0 Ad} \right) = \frac{1}{A} \frac{\partial Q}{\partial t} = \frac{I}{A} \]

6/4

Maxwell's eq

1. The divergence of curl is zero. $\nabla \cdot (\nabla \times E) = 0$.

2. From Ampere's Law, $\nabla \times B = \mu_0 J$

Drive $\nabla \cdot (\nabla \times B) = \nabla \cdot \mu_0 \left( \nabla + \varepsilon_0 \frac{\partial E}{\partial t} \right) = 0$

3. $\varepsilon_0 \frac{\partial E}{\partial t}$ is called as displacement current, if in the steady state

\[ \nabla \cdot j = -\frac{\partial P}{\partial t} \] continuity eq.
if \( \frac{E}{A} \) gives the current density, then the quantity \( \varepsilon_0 \frac{\partial E}{\partial t} \) can be interpreted as the density of the current. \( J = \varepsilon_0 \frac{\partial E}{\partial t}, \frac{E}{A}, \rho, \sigma \).

if a charging sphere

Battery: \( \frac{\partial E}{\partial t} + D \)

if the \( E \)-field is \( \frac{\varepsilon_0 (t)}{4\pi r^2} \)

\[
\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial E}{\partial t} \Rightarrow \frac{\partial E}{\partial t} = I, \quad 4\pi r^2 = A
\]

\[
\frac{\partial E}{\partial t} = \frac{I}{A \varepsilon_0} \Rightarrow \text{prove} \quad \varepsilon_0 \frac{\partial E}{\partial t} = \frac{I}{A} = J_D
\]

\( J_D \): Displacement current (density).

The vector function of the displacement current density \( J_d \) of curl must be zero.

\[
\nabla \times J_D = 0 = \varepsilon_0 \nabla \times \frac{\partial E}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times E)
\]

\[
\nabla \times B = \mu_0 J
\]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \Rightarrow \nabla \times J_0 = -\epsilon_0 \frac{3^2 B}{3^2 + 2} \]

**B-field changing with time.**

\[ E = \frac{-\gamma}{4\pi \varepsilon_0 r^2}, \quad \gamma = \text{constant} \]

(P.327)

\[ E(t, r) \]

\[ \Theta (v, t) = \Theta (v - r) \]

**Problem 7.34.** Suppose

\[ E(r, t) = \left(-\frac{1}{4\pi \varepsilon_0} \frac{\gamma}{r^2}\right) \left( \Theta (v - r) \right) \]

\[ \nabla \cdot E = \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z \]

\[ B(r, t) = 0 \]

Show that these fields satisfy all of the Maxwell's equations and determine \( J, J_0, J = pv \).

\[ \nabla \cdot B = 0, \quad \nabla \cdot E = 0 ? \]

\[ \nabla \times E = 0, \quad \nabla \times B = 0 \]

\[ \Rightarrow \text{Calculate } \nabla \cdot E \]

\[ \nabla \cdot \left( -\frac{\gamma}{4\pi \varepsilon_0} \frac{1}{r^2} \right) \left( \Theta (v - r) \right) \]

\[ = \Theta (v - r) \frac{\partial}{\partial r} \left( -\frac{\gamma}{4\pi \varepsilon_0} \frac{1}{r^2} \right) + \frac{\partial}{\partial r} \Theta (v - r) \]

\[ = \Theta (v - r) \frac{\partial}{\partial r} \left( -\frac{\gamma}{4\pi \varepsilon_0} \frac{1}{r^2} \right) + \Theta (v - r) \frac{\partial}{\partial r} \Theta (v - r) \]
The 1st term

\[ \Theta(u-t) \left( \frac{-q}{4 \pi \varepsilon_0} \right) \nabla \cdot \left( \frac{\delta}{r^2} \right) = \Theta(u-t) \left( \frac{-q}{4 \pi \varepsilon_0} \right) \left( 4 \pi \delta^3(r) \right) \]

\[ = \frac{-q}{4 \pi \varepsilon_0} \Theta(u-t) \delta^3(r) \]

The 2nd term

\[ \frac{-1}{4 \pi \varepsilon_0} \frac{q}{r^2} \left[ \nabla \cdot \nabla \left( \Theta(u-t) \right) \right] \]

\[ \Rightarrow 2^{nd} \text{ term} = \frac{q}{4 \pi \varepsilon_0 r^2} \delta(u-t) \]

\[ \nabla \cdot E = -\frac{q}{4 \pi \varepsilon_0} \delta^3(r) \Theta(t) + \frac{q}{4 \pi \varepsilon_0} \delta(u-t) = \frac{q}{\varepsilon_0} \]

\[ P = -q \delta^3(r) \Theta(t) + \frac{q}{4 \pi r^2} \delta(u-t) \]

\[ J_D = \varepsilon_0 \frac{\partial E}{\partial t} = \varepsilon_0 \left( \frac{-q}{4 \pi \varepsilon_0 r^2} \right) \frac{\partial}{\partial t} \delta(u-t) \]

\[ = \frac{-q}{4 \pi r^2} \nabla \delta(u-t) \]

\[ J = -\varepsilon_0 \frac{\partial E}{\partial t} \left( \Rightarrow \nabla \times B = \mu_0 J \right) \]

\[ \nabla \times E = \frac{\partial B}{\partial t} \]
3.3.4 Magnetic charge / Monopole

If has a magnetic charge density $\rho_m$

**Case I**  $\rho_e = 0$, $\rho_m = 0$, $J_e = 0$, $J_m = 0$

In free space, space-free

$\nabla \cdot E = 0$, $\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \cdot B = 0$, $\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

**Case II**  In free-space, But

Then we can replace

$\frac{\partial \vec{E}}{\partial t}$ by $\frac{\partial \vec{B}}{\partial t}$

$\frac{\partial \vec{B}}{\partial t}$ by $-\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

1. $\nabla \cdot \vec{E} = 0 \Rightarrow \nabla \cdot \vec{B} = 0$

2. $\nabla \cdot \vec{B} = 0 \Rightarrow \nabla \cdot (-\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = 0$

3. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{B} = -\frac{\partial}{\partial t} (-\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

4. $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\nabla \times (-\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = \mu_0 \varepsilon_0 \frac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

**Case III**. There are something missing

From $\nabla \cdot B = 0$, $\nabla \times E = -\frac{\partial B}{\partial t}$

if we has the two eqs for electric charges.

1. $\nabla \cdot E = \frac{\rho_e}{\varepsilon_0}$, $\nabla \times \vec{B} = \mu_0 \nabla \times \vec{E}$

2. $\nabla \cdot \vec{B} = \mu_0 \rho_m$  $\nabla \times \vec{E} = -\mu_0 \nabla \times \vec{B}$
\( P_m, P_e \equiv \) Define as the **density** of magnetic/electric charge

\( I_m, I_e \equiv \) Define as the **current** of magnetic/electric charge.

\& Both charges would be conserved.

\[
\nabla \cdot J_m = - \frac{\partial P_m}{\partial t}
\]

\[
\nabla \cdot J_e = - \frac{\partial P_e}{\partial t} \implies \nabla \cdot (J_m + J_e) = -\frac{\partial}{\partial t} (P_m + P_e) \implies ?
\]

Maxwell's eq. beg for the existence of magnetic charge. As far as we know, \( P_m \) is zero everywhere.

6/5 (III)

7.3.4 Magnetic charge

**continuity Eq. for magnetic charge**

\[
\nabla \cdot J_m = - \frac{\partial P_m}{\partial t} \quad L_m \cdot M_m \ldots
\]

**Problem 7.36**

Suppose a magnetic monopole \( J_m \) passes through a resistanceless loop of wire with self-inductance \( L \). What's the induced current in the loops.
From Faraday's Law

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

If there are one \( \Phi_m \), then

\[ \nabla \times \mathbf{E} = -\nabla \Phi_m - \frac{\partial \mathbf{B}}{\partial t} \quad \text{(modify Faraday's Law)} \]

\[ \int \nabla \times \mathbf{E} \, da = -\int \nabla \Phi_m \, da - \frac{\partial}{\partial t} \int \mathbf{B} \cdot da \]

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\nabla \Phi_m - \frac{\partial \Phi_m}{\partial t} = \mathcal{E} \quad \text{Eq of motion/circuit} \]

\[ \mathcal{E} = \mathcal{E}_m = \frac{\mathcal{E}_m}{A} \]

\[ \mathbf{F} = q \mathbf{E} \quad \int \mathbf{F} \cdot d\mathbf{l} = q \mathbf{w} \quad \int \frac{\mathbf{E}}{q} \cdot d\mathbf{l} = \mathcal{E} = \int \mathbf{F} \cdot d\mathbf{l} \]

If we know \( \mathcal{E} = -L \frac{dI}{dt} \)

\[ \Rightarrow -L \frac{dI}{dt} = -\mathcal{E}_m \cdot I - \frac{\partial \Phi}{\partial t} \]

\[ \frac{dI}{dt} = \frac{\mathcal{E}_m}{L} \frac{\partial \Phi}{\partial t} + \frac{l}{L} \frac{d\Phi}{dt} \]

Per unit \( \Phi \),

\[ \mathcal{I}_m = \frac{m \Phi_m}{L} + o \]
7.3.5 Maxwell's equations in Matter.

That an electric polarization $P$ produces magnetic magnetization $M$.

$$\text{bound charge } J_b = -\nabla \cdot P, \quad I$$

$$\text{bound current } J_b = \nabla \times M, \quad J \quad \text{(wrong)}$$

Any charge in the electric polarization involves a flow charge, which will be induced in the total current.

The polarized introduces a charge density $\sigma_b = P$ at one end $-\sigma_b$.

$$+ \sigma_b$$

If $P$ now increase a bit

$$dI = \frac{d\sigma_b}{dt} \, da \quad \text{at} \quad \frac{dP}{dt} \, da$$

The current density, therefore is

$$J_P = \frac{dP}{dt} ; \quad \therefore \nabla \cdot J_P = \nabla \cdot \frac{dP}{dt}$$

$$= \frac{d}{dt} \left[ \nabla \cdot P \right] = -\frac{dP}{dt}$$
The continuity eq. is satisfied in fact.

The total charge density \( \rho = \rho_f + \rho_b = \rho_f - \nabla \cdot P \)

\[ \text{current density} \quad J = J_f + J_b + J_L = \nabla \times M + \frac{\partial P}{\partial t} \]

So that the Gauss's law

\[ \nabla \cdot E = \frac{1}{\varepsilon_0} (\rho_f - \nabla \cdot P), \quad \nabla \cdot B = 0 \]

*So the Gauss's law in matter:

\[ \nabla \cdot \varepsilon_0 \varepsilon = \rho_f - \nabla \times P \]

\[ \Rightarrow \nabla \cdot (\varepsilon_0 \varepsilon + P) = \rho_f \]

\[ \nabla \cdot D = \rho_f \]  \[ \text{P.175 in the static state} \]

\[ D = \varepsilon_0 E + P \]

Meanwhile the Ampere's law in matter:

\[ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]  \[ \text{(in space)} \]

\[ \Rightarrow \nabla \times B = \mu_0 (J_f + \nabla \times M + \frac{\partial P}{\partial t}) + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]

\[ \nabla \times \frac{B}{\mu_0} = (J_f + \nabla \times M + \frac{\partial P}{\partial t}) + \varepsilon_0 \frac{\partial E}{\partial t} \]  \[ \text{(if} \quad H = \frac{B}{\mu_0} - M) \]

\[ \nabla \times \left( \frac{B}{\mu_0} - M \right) = J_f + \frac{\partial P}{\partial t} + \varepsilon_0 \frac{\partial E}{\partial t} \]

\[ \Rightarrow \nabla \times H = J_f + \frac{\partial P}{\partial t} \]  \[ \text{(\(\varepsilon_0 E + P = \nabla \times P\))} \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  \[ \text{Faraday's law} \]
**E. B in Matter**

Maxwell

1. \( \nabla \cdot \vec{B} = 0 \)
2. \( \nabla \cdot \vec{E} = 0 \)
3. \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)
4. \( \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \)


* 7.3.5 Magnetic monopole

For Gauss law for magnetic field \( \nabla \cdot \vec{B} = 0 \)?

Dirac showed the existence of monopole would explain electric charge is quantized.

magnetic charge, \( q = \frac{\hbar}{2me} \equiv 1.64 \times 10^{-9} \) A m


* Decay of free charge? "Important"

According to Maxwell's eq, a free charge should decay exponentially

\[ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \]

then

\[ \nabla \cdot (\nabla \times \vec{B}) = 0 \]

\[ \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \sigma \nabla \cdot \vec{E} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \nabla \cdot \vec{E} \]

\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \]
\[ \frac{M_0}{\varepsilon_0} \frac{\partial \rho}{\partial t} + \frac{M_0 \varepsilon_0}{\varepsilon_0} \rho = 0 \]

Integration results

\[ \rho = \rho_0 e^{-\frac{t}{\tau}}, \quad \tau = \frac{\varepsilon_0}{\varepsilon} \]

is known as the relaxation time.

The relation shows that any original distribution of charge decay exponentially at a rate that is independent of any other electromagnetic distribution.