13. The Revised Simplex Method

The LP \( \min c^T x, \text{ s.t. } Ax = b \geq 0 \), \( x \geq 0 \), can be represented by Figure 1(a) below. At any Simplex step, with known \( x_B \) and \( B^{-1} \), the Simplex tableau can be represented by Figure 1(b) below. The minimum is found if \( \bar{c}^T \geq 0 \).

![Simplex Tableau in Matrix Form](image)

**Remark.** Based on our convention, the z-row of the tableau is \(-c_B^T B^{-1} b\), negative of the actual objective function value. This is the consequence of defining reduced costs by \( c^T - c_B^T B^{-1} A \) as in the textbook.

Figure 1 suggests that knowing \( x_B \) and \( B^{-1} \) can generate the whole Simplex tableau, and hence can execute the Simplex method. Moreover, if there are simple rules to determine the new basic variables and to generate the new \( B^{-1} \), then the Simplex iterations can be carried out without keeping track of the whole Simplex tableau. The Revised Simplex method is such a procedure. Basically, it executes the exact Simplex method by keeping track of the change of the current basic variables \( x_B^{\text{cur}} \) to the new basic variables \( x_B^{\text{new}} \), and of the inverse of the current basis \( (B^{\text{cur}})^{-1} \) to the inverse of the new basis \( (B^{\text{new}})^{-1} \). Other than these steps, it detects the optimality and the unboundedness of an LP as the Simplex method does. In the following, we indicate how to execute the Simplex steps by the Revised Simplex method.

**Checking the Optimality Condition and Picking the Entering Variable:**

The reduced cost coefficient of any variable \( x_j \) is given by \( \bar{c}_j = c_j - c_B^T (B^{\text{cur}})^{-1} A_j \). In particular, check one by one \( \bar{c}_j \) for non-basic variable \( x_j \). If \( \bar{c}_j \geq 0 \) (\( \leq 0 \)) for all non-basic variable \( x_j \) for a minimization (maximization) problem, the current BFS is minimal (maximal). If not, pick any – in most cases the first – non-basic variable \( x_j \) with \( \bar{c}_j < (>) 0 \) as the
entering variable for a minimization (maximization) problem. By doing so, it is unnecessary to calculate all non-basic $c_j$ and apply the steepest ascending or descending rule. In any case, the rule does not guarantee to give the least number of iterations.

**Checking the Unboundedness Condition and Picking the Leaving Variable:**

Let $x_{enter}$ be the entering variable and $A_{enter}$ be its original column. With known inverse of the current basis $(B_{cur})^{-1}$, the column of $x_{enter}$ in the Simplex tableau, $A_{enter} = (B_{cur})^{-1}A_{enter}$. If all the entries of the column vector $A_{enter}$ are non-positive, the LP is unbounded and the Simplex iteration stops. Suppose the LP is bounded. Let the current RHS $b = (B_{cur})^{-1}b > 0$. Then $A_{enter}$ and $b$ provide inputs for the minimum ratio test to determine the entering variable.

**Updating $x_B$ and $c_B$:**

The new basic variable $x_{B,new}$ can be formed from the original basic variable $x_{B,cur}$ by replacing the leaving variable with the entering variable, and the new vector of the cost coefficients of the basic variables $c_{B,new}$ can be formed accordingly.

**Updating $B^{-1}$:**

A new $B^{-1}$ is formed by pivoting, which is equivalent to pre-multiplying by an elementary matrix. Suppose that $A_{enter} = \begin{pmatrix} a_{1,enter} \\ \vdots \\ a_{m,enter} \end{pmatrix}$ and the minimal ratio test says that $a_{i,leave,enter} (> 0)$ is the element to be pivoted on. The required pre-multiplying elementary matrix $E = [e_{i,j}]$ is found by changing the (leave)th column of the identity matrix $I$ such that $e_{i,leave} = \begin{cases} \frac{1}{a_{i,leave,enter}}, & \text{if } i = \text{leave}, \\ -\frac{\pi_{enter}}{a_{i,leave,enter}}, & \text{if } i \neq \text{leave}. \end{cases}$ Note that conceptually $(B_{cur})^{-1}$ is given by the columns in the tableau of the slack variables. The pivoting operations is equivalent to pre-multiplying $A$ with $E$, which turns the columns in the tableau of the slack variables into $(B_{new})^{-1}$. Thus,
$\left( B_{\text{new}} \right)^{-1} = E \left( B_{\text{cur}} \right)^{-1}.$

**Example 13.1.** Solve the LP \( \max 2x_1 + x_2, \ s.t. \ -x_1 + x_2 \leq 2, \ x_2 \leq 4, \ x_1 + x_2 \leq 8, \ x_1 \leq 6, \ x_1, x_2 \geq 0 \) by the revised Simplex method.

*Sol.* For benchmarking, we first solve the LP by the Simplex method, which we have turned the objective function into \( \min -2x_1 - x_2 \). Note that in this case the RHS is \(-c^T_B B^{-1} b\), the negative value of the actual objective function. Refer to the tableaus below. \( z^* = 14 \), with \( x_1^* = 6, \ x_2^* = 2, \ x_3^* = x_4^* = x_5^* = x_6^* = 0 \). Note that the variables \( x_3, x_4, x_5, \) and \( x_6 \) are slack variables added to the first to the fourth structural constraints, respectively.

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The following are the steps to solve the problem by the Revised Simplex method.

*First Iteration:* \( x_B = [x_3, x_4, x_5, x_6]^T \) and \( \left( B_{\text{cur}} \right)^{-1} = I \)
$$\mathbf{c}_B^T = [0, 0, 0, 0]. \ x_1 \text{ is non-basic. } \overline{c}_1 = c_1 - \mathbf{c}_B^T (\mathbf{B}_{\text{cur}})^{-1} \mathbf{A}_{\text{e}} = c_1 = -2. \ \text{The BFS is not minimal and } x_1 \text{ can be the entering variable.}$$

$$\overline{\mathbf{A}}_{\text{e}} = (\mathbf{B}_{\text{cur}})^{-1} \mathbf{A}_{\text{e}} = [-1, 0, 1, 1]^T. \ \text{The RHS is } (\mathbf{B}_{\text{cur}})^{-1} \mathbf{b} = [2, 4, 8, 6]^T. \ \text{The minimal ratio test gives } x_6 \text{ as the leaving variable. With } \overline{\mathbf{A}}_{\text{e}} = [-1, 0, 1, 1]^T \text{ and } x_6 \text{ the last basic variable leaving, } \mathbf{x}_{\text{B,new}} = [x_3, x_4, x_5, x_1]^T, (\mathbf{c}_{\text{B,new}})^T = [0, 0, 0, -2], \ \text{and } \mathbf{E}_1 =$$

$$= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

$$\begin{bmatrix}
1100 \\
0010 \\
0010 \\
0001
\end{bmatrix} \begin{bmatrix}
1001 \\
0100 \\
0011 \\
0001
\end{bmatrix} = \begin{bmatrix}
1001 \\
0100 \\
0011 \\
0001
\end{bmatrix}.$$

Second Iteration: $\mathbf{x}_B = [x_3, x_4, x_5, x_1]^T$ and $(\mathbf{B}_{\text{cur}})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}$

$$\mathbf{c}_B^T = [0, 0, 0, -2]. \ x_2 \text{ is non-basic. } \overline{\mathbf{A}}_{\text{e}} = (\mathbf{B}_{\text{cur}})^{-1} \mathbf{A}_{\text{e}} =$$

$$= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

$$\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

$$\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

To find the leaving variable, the current RHS $\overline{\mathbf{b}} = (\mathbf{B}_{\text{cur}})^{-1} \mathbf{b} =$$

$$= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
Third Iteration: \( x_B = [x_3, x_4, x_2, x_1]^T \) and \( (B_{\text{cur}})^{-1} \) =
\[
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( c_B^T = [0, 0, -1, -2] \). \( x_5 \) and \( x_6 \) are the non-basic variables. \( (B_{\text{cur}})^{-1} A_5 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \) and

\( (B_{\text{cur}})^{-1} A_6 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \) (Note. We can guess the values without multiplying out \( (B_{\text{cur}})^{-1} A_5 \) and \( (B_{\text{cur}})^{-1} A_6 \). Why? Hint. \( x_5 \) and \( x_6 \) are two of the slack variables that form the initial \( BFS \).)

\[
\bar{c}_5 = c_5 - c_B^T \overline{A}_{,5} = 0 - [0, 0, -1, -2] \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 1 \text{ and } \bar{c}_6 = c_6 - c_B^T \overline{A}_{,6} = -[0, 0, -1, -2] \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 1.
\]

Since the reduced cost coefficients of all the non-basic variables are non-negative (in fact, positive), the minimal solution is got. \( x_B^* = \begin{bmatrix} x_3^* \\ x_4^* \\ x_2^* \\ x_1^* \end{bmatrix} = (B_{\text{cur}})^{-1} b = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 2 \\ 6 \end{bmatrix} \), \( x_5^* = x_6^* = 0 \), \( z^* = c_{B_{\text{cur}}}^T (B_{\text{cur}})^{-1} b = [0, 0, 1, 2] \begin{bmatrix} 6 \\ 2 \\ 2 \\ 6 \end{bmatrix} = 14. \)

Check that the intermediate and final results of the Revised Simplex method are exactly the same as those of the Simplex method.