

Separable Programming

Refer to the following example in which we solve an optimization problem with non-linear objective function and linear constraints by a linear program. Later we will extend the idea to solve, at least approximately, convex optimization problems with separable non-linear objective functions and convex feasible sets defined by separable convex functions. The technique, called *separable programming*, basically replaces all separable convex functions, in objectives and constraints, by piecewise linear convex functions.

Example 1. (Example 1.4.2 of Prof. Murty's notes MS-1.pdf): A company makes products P1, P2, P3 using limestone (LI), electricity (EP), water (W), fuel (F), and labor (L) as inputs. Labor is measured in manhours, other inputs in suitable units. Each input is available from one or more sources. The company has its own quarry for LI, which can supply up to 250 units/day at a cost of \$20/unit. Beyond that, LI can be purchased in any amounts from an outside supplier at \$50/unit. EP is only available from the local utility. Their charges for EP are: \$30/unit for the first 1000 units/day, \$45/unit for up to an additional 500 units/day beyond the initial 1000units/day, \$75/unit for amounts beyond 1500 units/day. Up to 800 units/day of W (water) is available from the local utility at \$6/unit, beyond that they charge \$7/unit of water/day. There is a single supplier for F who can supply at most 3000 units/day at \$40/unit, beyond that there is currently no supplier for F. From their regular workforce they have up to 640 manhours of labor/day at \$10/manhour, beyond that they can get up to 160 manhours/day at \$17/manhour from a pool of workers. They can sell up to 50 units of P1 at \$3000/unit/day in an upscale market; beyond that they can sell up to 50 more units/day of P1 to a wholesaler at \$250/unit. They can sell up to 100 units/day of P2 at \$3500/unit. They can sell any quantity of P3 produced at a constant rate of \$4500/unit. Data on the inputs needed to make the various products is given in the following table. Formulate the product mix problem to maximize the net profit/day at this company.

Product	Input units/unit of output				
	LI	EP	W	F	L
P1	1/2	3	1	1	2
P2	1	2	1/4	1	1
P3	3/2	5	2	3	1

Sol. Let P_i be the number of units that product i is produced, $i = 1, 2, 3$; LI, EP, W, F, and L be the numbers of units limestone, electricity, water, fuel, and labor used to produce the products. Then the problem can be formulated as the non-linear optimization problem **OP1**:

$$\begin{aligned} \text{Min } z = & \max(20LI, 50LI-7500) + \max(30EP, 45EP-15000, 75EP-60000) \\ & + \max(6W, 7W-1800) + 40F + \max(10L, 17L-4480) \\ & - \min(3000P_1, 250P_1+137500) - 3500P_2 - 4500P_3 \end{aligned}$$

subject to

$$(1/2)P_1 + P_2 + (3/2)P_3 = LI \quad (\text{limestone used by products})$$

$$3P_1 + 2P_2 + 5P_3 = EP \quad (\text{electricity used by products})$$

$$P_1 + (1/4)P_2 + 2P_3 = W \quad (\text{water used by products})$$

$$P_1 + P_2 + 3P_3 = F \quad (\text{fuel used by products})$$

$$2P_1 + P_2 + P_3 = L \quad (\text{labor used by products})$$

$$\text{All variables} \geq 0$$

$$(F, P_1, P_2) \leq (3000, 100, 100).$$

Check that $\max(\cdot, \cdot)$, $\max(\cdot, \cdot, \cdot)$, and $-\min(\cdot, \cdot)$ are all convex functions. The problem is indeed a convex optimization problem. By defining variables $y_{LI} = \max(20LI, 50LI-7500)$, $y_{EP} = \max(30EP, 45EP-15000, 75EP-60000)$, $y_W = \max(6W, 7W-1800)$, $y_L = \max(10L, 17L-4480)$, and $y_{P1} = \min(3000P_1, 250P_1+137500)$, the problem can be formulated as the following linear program **OP2**.

$$\text{Min } z = y_{LI} + y_{EP} + y_W + 40F + y_L - y_{P1} - 3500P_2 - 4500P_3$$

subject to

$$20LI \leq y_{LI}$$

$$50LI-7500 \leq y_{LI}$$

$$30EP \leq y_{EP}$$

$$45EP-15000 \leq y_{EP}$$

$$75EP-60000 \leq y_{EP}$$

$$6W \leq y_W$$

$$7W-1800 \leq y_W$$

$$10L \leq y_L$$

$$17L-4480 \leq y_L$$

$$y_{P1} \leq 3000P_1$$

$$y_{P1} \leq 250P_1+137500$$

$$(1/2)P_1 + P_2 + (3/2)P_3 = LI \quad (\text{limestone used by products})$$

$$3P_1 + 2P_2 + 5P_3 = EP \quad (\text{electricity used by products})$$

$$P_1 + (1/4)P_2 + 2P_3 = W \quad (\text{water used by products})$$

$$P_1 + P_2 + 3P_3 = F \quad (\text{fuel used by products})$$

$$2P_1 + P_2 + P_3 = L \quad (\text{labor used by products})$$

$$\text{All variables} \geq 0$$

$$(F, P_1, P_2) \leq (3000, 100, 100).$$

As we know, there is another formulation that can turn **OP1** into a linear program. Let LI_1 and LI_2 be the numbers of units of limestone obtained in its two prices; W_1 and W_2 be the numbers of units of water obtained in its two prices; EP_1 , EP_2 , and EP_3 be the numbers of units of electricity obtained in its three prices; L_1 and L_2 be the numbers of units of labor obtained in its two prices; and P_{11} and P_{12} be the numbers of units of product 1 sold in its two prices. Then **OP1** can be reformulated as **OP3**.

$$\text{Min } z = 20LI_1 + 50LI_2 + 30EP_1 + 45EP_2 + 75EP_3 + 6W_1 + 7W_2 + 40F + 10L_1 + 17L_2 \\ - 3000P_{11} - 250P_{12} - 3500P_2 - 4500P_3$$

subject to

$$\begin{aligned} (1/2)P_1 + P_2 + (3/2)P_3 &= LI && \text{(limestone used by products)} \\ 3P_1 + 2P_2 + 5P_3 &= EP && \text{(electricity used by products)} \\ P_1 + (1/4)P_2 + 2P_3 &= W && \text{(water used by products)} \\ P_1 + P_2 + 3P_3 &= F && \text{(fuel used by products)} \\ 2P_1 + P_2 + P_3 &= L && \text{(labor used by products)} \\ LI_1 + LI_2 &= LI && \text{(limestone obtained in its 2 prices)} \\ W_1 + W_2 &= W && \text{(water obtained in its 2 prices)} \\ EP_1 + EP_2 + EP_3 &= EP && \text{(electricity obtained in its 3 prices)} \\ L_1 + L_2 &= L && \text{(labor obtained in its 2 prices)} \\ P_{11} + P_{12} &= P_1 && \text{(product 1 sold in its 2 prices)} \\ \text{All variables} &\geq 0 \\ (LI_1, EP_1, EP_2, W_1) &\leq (250, 1000, 500, 800) \\ (F, L_1, L_2) &\leq (3000, 640, 160) \\ (P_{11}, P_{12}, P_2) &\leq (50, 50, 100). \end{aligned}$$

We will see how to extend the above idea to general non-linear separable convex objective functions and constraints defined by non-linear separable convex functions.

Fact 2. Any (continuous) convex function can be approximated to any degree of accuracy by a piecewise linear convex function.

Let $f_i(x_i)$ be a convex function for every i . The separable program

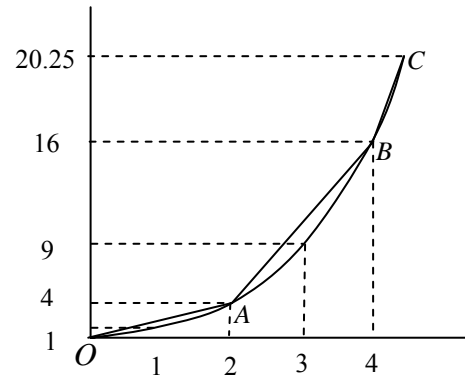
$$\min \sum_i f_i(x_i) \text{ subject to linear constraints}$$

is in fact a convex program. Any local is indeed a global minimum. This property preserves if we approximate each f_i by a piecewise linear function. From Fact 2, doing so

can lead to approximate minimum objective function value and approximate minimum solution that are arbitrarily close to the respective true values.

Example 3. **NLP1** is:

$$\begin{aligned} \min \quad & x_1^2 - 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 5, \\ & 2x_1 + x_2 \leq 9, \\ & x_1, x_2 \geq 0. \end{aligned}$$



The above is a separable program with $f_1(x_1) = x_1^2 - 2x_1$ and $f_2(x_2) = -x_2$. f_2 is linear. The problem is easy to solve if f_1 , i.e., x_1^2 is approximated by a linear function.

From the constraints, $x_1 \leq 4.5$. Let us approximate the non-linear function $y = x_1^2$ by a piecewise linear function. For simplicity, we take a function of three linear pieces, with break points at 0, 2, 4, and 4.5. The same procedure can be applied to any number of break points at any values.

points	<i>O</i>	<i>A</i>	<i>B</i>	<i>C</i>
x_1	0	2	4	4.5
y	0	4	16	20.25

How to represent the function of three linear pieces? There are two ways, the λ - and the δ -forms.

λ -form: For any point within a linear segment, its functional value is the convex combination of the values of the two break points of the linear segment. Let $\lambda_i \geq 0$ be the weight of break point i , $i = O, A, B$, and C .

$$\left\{ \begin{aligned} x_1 &= 0\lambda_O + 2\lambda_A + 4\lambda_B + 4.5\lambda_C, & (1) \\ y &= 0\lambda_O + 4\lambda_A + 16\lambda_B + 20.25\lambda_C, & (2) \\ \lambda_O + \lambda_A + \lambda_B + \lambda_C &= 1, & (3) \\ \text{at most two adjacent } \lambda_i &\text{ take non - zero values.} & (4) \end{aligned} \right.$$

Now reformulate **NLP1** into

NLP2

$$\begin{aligned}
 \min \quad & y - 2x_1 - x_2 \\
 \text{s.t.} \quad & x_1 + 2x_2 \leq 5, \\
 & 2x_1 + x_2 \leq 9, \\
 & (1), (2), (3), \text{ and } (4), \\
 & x_1, x_2, \lambda_i, i = O, A, B, C \geq 0.
 \end{aligned}
 \quad \diamond \diamond$$

Constraints (4) can be omitted if we start with a convex program. Check that whenever (4) is violated by a set of λ_i , the value of the corresponding y is above the three-piece linear function, and hence the set of λ_i cannot be a minimum point.

In general, if **NLP1** is not a convex program, constraints (4) are needed, though they are implemented implicitly through the separable programming extension of the simplex method. The pivoting step of the separable programming extension simply ensures that at any time at most two adjacent λ_i are in the basis. It can be shown that such a restricted basis rule will lead to the optimum solution.

The idea of separable program is applicable to non-convex programs. Of course, in those cases, the optimum solution can be a local rather than global optimum.

δ -form: Instead of taking weighted value of break points, we can add up the contribution from each linear segment. Let δ_i be the proportion of the i th segment taken, $i = OA, AB, BC$.

$$\left\{ \begin{array}{l}
 x_1 = 2\delta_{OA} + 2\delta_{AB} + 0.5\delta_{BC}, \quad (5) \\
 y = 4\delta_{OA} + 12\delta_{AB} + 4.25\delta_{BC}, \quad (6) \\
 0 \leq \delta_{OA}, \delta_{AB}, \delta_{BC} \leq 1, \quad (7) \\
 \text{there exists } \delta_j > 0 \text{ such that } \delta_i = 1 \text{ for } i < j \text{ and } \delta_i = 0 \text{ for } i > j. \quad (8)
 \end{array} \right.$$

As for the λ -form, the δ -form may also give a local optimum if the original program is non-convex. When the original program is convex, constraint (8) is not necessary.

Remark 4. To get more accurate result, the piecewise linear approximation of f_i can be

refined with more linear segments. There are studies on segment refinement to get the best trade off between accuracy and computational effort.

Remark 5. It is possible to approximate constraints by similar procedure.

Remark 6. A product term x_1x_2 can be transformed to separable form. Let $s_1 = (x_1+x_2)/2$ and $s_2 = (x_1-x_2)/2$. Then $x_1x_2 = s_1^2 - s_2^2$.