Probability Theory
Instructor: Yu-Ling Tseng Homework due on 1228 20171214

1. For $n = 1, 2, 3, \ldots$, let X_n be independent r.v.'s such that

$$P(X_n = 1) = p_n, \quad P(X_n = 0) = 1 - p_n.$$

Under what conditions on the p_n's, as $n \longrightarrow \infty$, does $X_n \xrightarrow{p} 0$?

2. Let X_n, $n = 1, 2, \ldots$, be i.i.d. r.v.'s such that $E(X_n) = \mu$, $Var(X_n) = \sigma^2$, both finite. Show that $E(\bar{X} - \mu)^2 \longrightarrow 0$, that is $\bar{X} \xrightarrow{q.m.} \mu$, as $n \longrightarrow \infty$.

3. For $n = 1, 2, \ldots$, let X_n, Y_n be r.v.'s such that, as $n \longrightarrow \infty$, $E(X_n - Y_n)^2 \longrightarrow 0$ and suppose that $X_n \xrightarrow{q.m.} X$ for some r.v. X. Then show that $Y_n \xrightarrow{q.m.} X$, as $n \longrightarrow \infty$.

4. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. $U(0, 1)$ r.v.'s, and for each n, set $Y_n = \min\{X_1, \ldots, X_n\}$, $Z_n = \max\{X_1, \ldots, X_n\}$, $U_n = nY_n$, $V_n = n(1 - Z_n)$. Show that, as $n \longrightarrow \infty$, one has
   (a) $Y_n \xrightarrow{p} 0$;
   (b) $Z_n \xrightarrow{p} 1$;
   (c) $U_n \xrightarrow{d} U$;
   (d) $V_n \xrightarrow{d} U$, where $U \sim f(u) = e^{-u}, \quad u > 0$, a (negative) exponential distribution with parameter 1.

5. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. r.v.'s and suppose that the k-th population moment $E(X_1^k) = \theta_k$ is finite for a given positive integer k. For $n = 1, 2, \ldots$, let

$$m_k(X) = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

be the k-th sample moment of the X_i's, show that $m_k(X) \xrightarrow{p} \theta_k$, as $n \longrightarrow \infty$. Thus, show that for i.i.d. r.v.'s with finite variance, the sample variance converges in probability to the population variance.