Power Estimation for Testing Normal Means

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Outline

Hypothesis Tests

- P-value : data-sensitive "evidence" against the null

  Berger & Sellke , Berger & Delampady, Casella & Berger '87

  $*$ p-value and the Bayes estimates are irreconcilable, two-sided / reconcilable, one-sided
- Observed power : strength of the experiment

  Power analysis:

  $*$ not small p-value + high observed power

  $\longrightarrow$ strong evidence supporting null hypothesis.

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  $\longrightarrow$ results: significant & test: very powerful

  Gillett '96, Hoening & Heisey, Lenth 2001

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$\hookrightarrow$ Q: Validity of such power analysis? How?

Basic settings

- Model: $X_1, \cdots, X_n$ iid $\sim N(\theta, \sigma^2)$, $\sigma^2 > 0$: known
- Testing problems:

$$H_0 : \theta \leq 0 \quad vs. \quad H_1 : \theta > 0, \tag{1}$$

$$H_0 : \theta = 0 \quad vs. \quad H_1 : \theta \neq 0. \tag{2}$$

- The UMP α level test for (1) is to reject H_0 if

$$\frac{\bar{X}}{\sigma_n} > z_\alpha, \qquad \text{power ft} \quad \beta_1(\theta) = \Phi\left(\frac{\theta}{\sigma_n} - z_\alpha\right) \tag{3}$$

The UMPU α level test for (2) is to reject H_0 if

$$\frac{|\bar{X}|}{\sigma_n} > z_{\alpha/2}, \; \beta_2(\theta) = \Phi\left(\frac{\theta}{\sigma_n} - z_{\alpha/2}\right) + \Phi\left(\frac{-\theta}{\sigma_n} - z_{\alpha/2}\right) \tag{4}$$

w/ $\sigma_n^2 = \sigma^2/n$, $\Phi(z_\alpha) = 1 - \alpha$, $\Phi(x)$ cdf of $N(0,1)$

Observed powers

The usual *observed powers* are

$$\beta_1(\hat{\theta}) = \Phi\left(\frac{\hat{\theta}}{\sigma_n} - z_\alpha\right) = P_{\hat{\theta}}\left(\frac{\bar{X}}{\sigma_n} > z_\alpha\right) \tag{5}$$

$$\begin{aligned}\beta_2(\hat{\theta}) &= \Phi\left(\frac{\hat{\theta}}{\sigma_n} - z_{\alpha/2}\right) + \Phi\left(\frac{-\hat{\theta}}{\sigma_n} - z_{\alpha/2}\right) \\ &= P_{\hat{\theta}}\left(\frac{|\bar{X}|}{\sigma_n} > z_{\alpha/2}\right)\end{aligned} \tag{6}$$

w/ $\hat{\theta} = \bar{x} = \sum_i x_i / n$ MLE of θ

$\hookrightarrow$ observed powers are MLE of the powers.

A key reduction

Dalal & Hall '83, Tsao 2006

Lemma

Let X be normally distributed and let $\beta(\theta)$ be a bounded and integrable function then for any $\pi(\theta) \in \Gamma_{BCPS}$,

$$\sup_{\pi \in \Gamma_{BCPS}} E_{\pi(\theta|x)} \beta(\theta) = \sup_{\pi \in \Gamma_{NORS}} E_{\pi(\theta|x)} \beta(\theta). \tag{7}$$

where

$$\Gamma_{NORS} = \{\pi | \pi = \frac{1}{2}(\pi_+ + \pi_-)\ w/\ \pi_+ \sim N(\mu, \tau^2), \pi_- \sim N(-\mu, \tau^2)\}.$$

One-sided problems

For $\bar{x} > 0$, and define $a \vee b = \max(a, b)$.

- $$\sup_{\pi \in \Gamma_{BCPS}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) = \sup_{\pi \in \Gamma_{N(0,\tau^2)}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) \vee 1/2,$$
- $\pi \in \Gamma_{N(0,\tau^2)} \implies E_{\pi(\theta|\bar{x})} \beta_1(\theta)$ increases in τ^2.
- $$\sup_{\pi \in \Gamma_{N(0,\tau^2)}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) = \lim_{\pi \sim N(0,\tau^2), \tau^2 \to \infty} E_{\pi(\theta|\bar{x})} \beta_1(\theta) \quad (8)$$
  $$= E_{L(\theta|\bar{x})} \beta_1(\theta) \quad (9)$$

  w/ $L(\theta|\bar{x})$ is the likelihood function given $\bar{X} = \bar{x}$.

Setting for simulations

- $a = a_1,\ a_2,\ a_3,\ a_4,\ a_5$ and $\beta = \beta_0,\ \beta_1,\ \beta_2,\ \beta_3$
- $r = 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 25$ (Hwang and Casella, 1982)
- $\alpha = 0.25,\ 0.1,\ 0.05$
- Simulation number: $M = 10^8$

Table:

Notation	Dimension	Notation	Dimension
— ○ —	r=3	—◊—	r=11
— △ —	r=5	—▽—	r=13
— + —	r=7	— ⊠ —	r=15
— × —	r=9	— ∗ —	r=25

Table: Theoretical values of D_*.

		$\alpha = 0.05$		$\alpha = 0.1$	
p	B	A	D_*	A	D_*
3	1	9.52	49.208	7.43	45.849
4	0.5	6.27	52.228	4.99	49.753
5	$1/3$	5.09	57.159	4.20	55.542
6	0.25	4.46	64.588	3.79	60.984

Note: B is taken to be $\frac{1}{p-2}$ here.

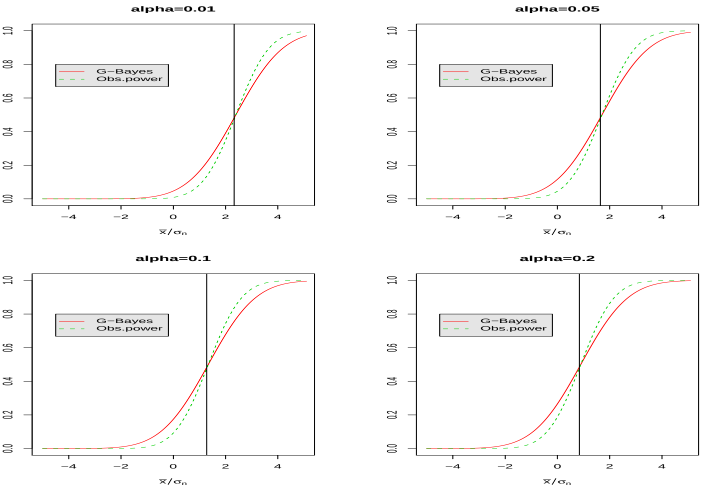

Figure: Observed power versus generalized Bayes estimate: 1-sided problem

Other theoretical results

The limiting cases. $(m \longrightarrow \infty)$

- Consider normal priors and let $w_\infty(t) = \lim_{m\to\infty} w(t)$, then

$$\lim_{\tau^2\to\infty} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_\infty(t) = 1-\alpha,$$

$$\lim_{\mu\to 0} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_\infty(t) = \Phi\left[\rho^{1/2}\left((\frac{\sigma_n^2}{\sigma_n^2+\tau^2})\bar{x} + \sigma_n z_\alpha\right)\right]$$

Conclusions

- We study the post-data performance of one-sided normal tolerance intervals.
  * conf. coeff. tends to be more extreme than Bayes est.'s
  * discrepancy is more marked as sample size n increases.
- Our result also hints a way to choose/construct the prior or mixing distribution in the de Finetti's representation theorem.
  * The practtice of using beta prior as the "natural" priors for $0-1$ r.v.'s, in this context, is justifiable since the derived $\lambda(t|\mu,\tau^2;\bar{x})$ can be well-approximated by a beta distribution.
  * Nonetheless, $\lambda(t|\mu,\tau^2;\bar{x})$ has better analytical tractability.
- Further research in unknown variance and two-sided tolerance intervals are of importance yet demands more involved calculations.

Thanks for your attention!