Power Estimation for Testing Normal Means

C. Andy Tsao   Yu-Ling Tseng*
National Dong Hwa University, Taiwan

ISI 2007 at Lisboa, Portugal
Outline

Introduction

The Problem Formulation

Theoretical Results

Numerical Results

Conclusions
Hypothesis Tests

- **P-value**: data-sensitive “evidence” against the null
  
  Berger & Sellke, Berger & Delampady, Casella & Berger ’87
  
  * p-value and the Bayes estimates are irreconcilable, two-sided / reconcilable, one-sided

- **Observed power**: strength of the experiment

  Power analysis:
  
  * not small p-value + high observed power
    
    → strong evidence supporting null hypothesis.
  
  * small p-value + high observed power
    
    → results: significant & test: very powerful

Gillett ’96, Hoening & Heisey, Lenth 2001
Hypothesis Tests

- **P-value**: data-sensitive “evidence” against the null
  - Berger & Sellke, Berger & Delampady, Casella & Berger ’87
  - *p-value and the Bayes estimates are irreconcilable, two-sided / reconcilable, one-sided*

- **Observed power**: strength of the experiment
  - Power analysis:
    - *not small p-value + high observed power* → strong evidence supporting null hypothesis.
    - *small p-value + high observed power* → results: significant & test: very powerful
  
  Gillett ’96, Hoening & Heisey, Lenth 2001

→ **Q**: Validity of such power analysis? How?
Basic settings

- Model: $X_1, \cdots, X_n$ iid $\sim N(\theta, \sigma^2)$, $\sigma^2 > 0$: known
- Testing problems:

\[
H_0 : \theta \leq 0 \quad \text{vs.} \quad H_1 : \theta > 0, \quad (1)
\]
\[
H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta \neq 0. \quad (2)
\]

- The UMP $\alpha$ level test for (1) is to reject $H_0$ if

\[
\frac{\bar{X}}{\sigma_n} > z_\alpha, \quad \text{power ft} \quad \beta_1(\theta) = \Phi \left( \frac{\theta}{\sigma_n} - z_\alpha \right) \quad (3)
\]

The UMPU $\alpha$ level test for (2) is to reject $H_0$ if

\[
\frac{|\bar{X}|}{\sigma_n} > z_{\alpha/2}, \quad \beta_2(\theta) = \Phi \left( \frac{\theta}{\sigma_n} - z_{\alpha/2} \right) + \Phi \left( -\frac{\theta}{\sigma_n} - z_{\alpha/2} \right) \quad (4)
\]

w/ $\sigma_n^2 = \sigma^2 / n$, $\Phi(z_\alpha) = 1 - \alpha$, $\Phi(x)$ cdf of $N(0, 1)$
Observed powers

The usual \textit{observed powers} are

\begin{align}
\beta_1(\hat{\theta}) &= \Phi \left( \frac{\hat{\theta}}{\sigma_n} - z_\alpha \right) = P_{\hat{\theta}} \left( \frac{\bar{X}}{\sigma_n} > z_\alpha \right) \\
\beta_2(\hat{\theta}) &= \Phi \left( \frac{\hat{\theta}}{\sigma_n} - z_\alpha/2 \right) + \Phi \left( \frac{-\hat{\theta}}{\sigma_n} - z_\alpha/2 \right) \\
&= P_{\hat{\theta}} \left( \frac{|\bar{X}|}{\sigma_n} > z_\alpha/2 \right)
\end{align}

w/ $\hat{\theta} = \bar{x} = \sum_i x_i / n$ \hspace{1cm} \text{MLE of } \theta

\rightarrow \text{observed powers are MLE of the powers.}
A key reduction

Dalal & Hall ’83, Tsao 2006

Lemma

Let $X$ be normally distributed and let $\beta(\theta)$ be a bounded and integrable function then for any $\pi(\theta) \in \Gamma_{BCPS},$

$$\sup_{\pi \in \Gamma_{BCPS}} E_{\pi(\theta|x)} \beta(\theta) = \sup_{\pi \in \Gamma_{NORS}} E_{\pi(\theta|x)} \beta(\theta). \quad (7)$$

where

$$\Gamma_{NORS} = \{ \pi | \pi = \frac{1}{2} (\pi_+ + \pi_-) \text{ w/ } \pi_+ \sim N(\mu, \tau^2), \pi_- \sim N(-\mu, \tau^2) \}. $$
One-sided problems

For \( \bar{x} > 0 \), and define \( a \lor b = \max(a, b) \).

\[
\sup_{\pi \in \Gamma_{BCPS}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) = \sup_{\pi \in \Gamma_{N(0, \tau^2)}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) \lor 1/2,
\]

\[
\pi \in \Gamma_{N(0, \tau^2)} \implies E_{\pi(\theta|\bar{x})} \beta_1(\theta) \text{ increases in } \tau^2.
\]

\[
\sup_{\pi \in \Gamma_{N(0, \tau^2)}} E_{\pi(\theta|\bar{x})} \beta_1(\theta) = \lim_{\pi \sim N(0, \tau^2), \tau^2 \to \infty} E_{\pi(\theta|\bar{x})} \beta_1(\theta) \quad (8)
\]

\[
= E_L(\theta|\bar{x}) \beta_1(\theta) \quad (9)
\]

w/ \( L(\theta|\bar{x}) \) is the likelihood function given \( \bar{X} = \bar{x} \).
Setting for simulations

- \( a = a_1, a_2, a_3, a_4, a_5 \) and \( \beta = \beta_0, \beta_1, \beta_2, \beta_3 \)
- \( r = 3, 5, 7, 9, 11, 13, 15, 25 \) (Hwang and Casella, 1982)
- \( \alpha = 0.25, 0.1, 0.05 \)
- Simulation number: \( M = 10^8 \)

Table:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Dimension</th>
<th>Notation</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>- ○ -</td>
<td>r=3</td>
<td>- ♦ -</td>
<td>r=11</td>
</tr>
<tr>
<td>- △ -</td>
<td>r=5</td>
<td>- ▽ -</td>
<td>r=13</td>
</tr>
<tr>
<td>- + -</td>
<td>r=7</td>
<td>- ◯ -</td>
<td>r=15</td>
</tr>
<tr>
<td>- × -</td>
<td>r=9</td>
<td>- * -</td>
<td>r=25</td>
</tr>
</tbody>
</table>
Table: Theoretical values of $D_\ast$.

<table>
<thead>
<tr>
<th>p</th>
<th>B</th>
<th>A</th>
<th>$D_\ast$</th>
<th>A</th>
<th>$D_\ast$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>9.52</td>
<td>49.208</td>
<td>7.43</td>
<td>45.849</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>6.27</td>
<td>52.228</td>
<td>4.99</td>
<td>49.753</td>
</tr>
<tr>
<td>5</td>
<td>$1/3$</td>
<td>5.09</td>
<td>57.159</td>
<td>4.20</td>
<td>55.542</td>
</tr>
<tr>
<td>6</td>
<td>0.25</td>
<td>4.46</td>
<td>64.588</td>
<td>3.79</td>
<td>60.984</td>
</tr>
</tbody>
</table>

Note: $B$ is taken to be $\frac{1}{p-2}$ here.
Figure: Observed power versus generalized Bayes estimate: 1-sided problem

C. Andy Tsao, Yu-Ling Tseng* National Dong Hwa University, Taiwan
Power Estimation for Testing Normal Means
Other theoretical results

The limiting cases. \((m \rightarrow \infty)\)

- Consider normal priors and let \(w_\infty(t) = \lim_{m \rightarrow \infty} w(t)\), then

\[
\lim_{\tau^2 \rightarrow \infty} E_{\lambda(t|\mu, \tau^2; \bar{x})} w_\infty(t) = 1 - \alpha,
\]

\[
\lim_{\mu \rightarrow 0} E_{\lambda(t|\mu, \tau^2; \bar{x})} w_\infty(t) = \Phi \left[ \rho^{1/2} \left( \frac{\sigma^2}{\sigma^2_n + \tau^2} \bar{x} + \sigma_n z_\alpha \right) \right]
\]
Conclusions

- We study the post-data performance of one-sided normal tolerance intervals.
  * conf. coeff. tends to be more extreme than Bayes est.’s
  * discrepancy is more marked as sample size $n$ increases.

- Our result also hints a way to choose/construct the prior or mixing distribution in the de Finetti’s representation theorem.
  * The practice of using beta prior as the “natural” priors for $0 - 1$ r.v.’s, in this context, is justifiable since the derived $\lambda(t|\mu, \tau^2; \bar{x})$ can be well-approximated by a beta distribution.
  * Nonetheless, $\lambda(t|\mu, \tau^2; \bar{x})$ has better analytical tractability.

- Further research in unknown variance and two-sided tolerance intervals are of importance yet demands more involved calculations.
Thanks for your attention!