Test 2 for the Course of Management Mathematics

Date: Jan. 2, 2007 Time: 9:30-11:30

1. The Simplex Method: (25%)

Scheduling Products A, B, and C are sold door-to-door. Product A costs \$3 per unit, takes 10 minutes to sell (on the average), and costs \$0.50 to deliver to the customer. Product B costs \$5, takes 15 minutes to sell, and is left with the customer at the time of sale. Product C costs \$4, takes 12 minutes to sell, and costs \$1.00 to deliver. During any week a salesperson is allowed to draw up to \$500 worth of A, B, and C (at cost) and is allowed delivery expenses not to exceed \$75. If a salesperson's selling time is not expected to exceed 30 hours (1800 minutes) in a week, and if the salesperson's profit (net after all expenses) is \$1 each on a unit of A or B and \$2 on a unit of C, what combination of sales of A, B, and C will lead to maximum profit and what is this maximum profit?

We let x_1, x_2, and x_3 represent the number of products A, B, and C sold. We want to

Maximize $P = x_1 + x_2 + 2x_3$

subject to the following constraints

$$\begin{array}{ll}
3x_1 + 5x_2 + 4x_3 \le 500 & \text{cost limit of products} \\
10x_1 + 15x_2 + 12x_3 \le 1800 & \text{time available for selling} \\
.50x_1 + x_3 \le 75 & \text{amount available for delivery} \\
x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0 &
\end{array}$$

Slack variables are added and the initial tableau is constructed and shown below. The pivot element is 1.

$$\begin{array}{c} \text{BV} \\ s_1 \\ s_2 \\ \rightarrow s_3 \\ P \end{array}
\begin{array}{c} \begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & 3 & 5 & 4 & 1 & 0 & 0 & 500 \\
0 & 10 & 15 & 12 & 0 & 1 & 0 & 1800 \\
0 & .5 & 0 & \boxed{1} & 0 & 0 & 1 & 75 \\ \hline
1 & -1 & -1 & -2 & 0 & 0 & 0 & 0
\end{array}\right] \end{array}
\longrightarrow
\begin{array}{c} \text{BV} \\ s_1 \\ s_2 \\ \rightarrow x_3 \\ P \end{array}
\begin{array}{c} \begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & 1 & 5 & 0 & 1 & 0 & -4 & 200 \\
0 & 4 & 15 & 0 & 0 & 1 & -12 & 900 \\
0 & .5 & 0 & \boxed{1} & 0 & 0 & 1 & 75 \\ \hline
1 & 0 & -1 & 0 & 0 & 0 & 2 & 150
\end{array}\right] \end{array}$$

Since there is still a negative entry in the objective row, it denotes the new pivot column. The pivot row is determined by choosing the smallest quotient when the RHS is divided by the elements in the pivot column. The new pivot element is 5.

$$\begin{array}{c} \text{BV} \\ \rightarrow s_1 \\ s_2 \\ x_3 \\ P \end{array}
\begin{array}{c} \begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & 1 & \boxed{5} & 0 & 1 & 0 & -4 & 200 \\
0 & 4 & 15 & 0 & 0 & 1 & -12 & 900 \\
0 & .5 & 0 & 1 & 0 & 0 & 1 & 75 \\ \hline
1 & 0 & -1 & 0 & 0 & 0 & 2 & 150
\end{array}\right] \end{array}
\longrightarrow
\begin{array}{c} \text{BV} \\ \rightarrow x_2 \\ s_2 \\ x_3 \\ P \end{array}
\begin{array}{c} \begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & \frac{1}{5} & \boxed{1} & 0 & \frac{1}{5} & 0 & -\frac{4}{5} & 40 \\
0 & 1 & 0 & 0 & -3 & 1 & 0 & 300 \\
0 & .5 & 0 & 1 & 0 & 0 & 1 & 75 \\ \hline
1 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & \frac{6}{5} & 190
\end{array}\right] \end{array}$$

Since all the entries in the objective row are nonnegative, this is the final tableau. The solution is $P = 190$, obtained when $x_1 = 0$, $x_2 = 40$, and $x_3 = 75$.

So the salesperson maximizes profit by selling no product A, 40 units of product B an 75 units of product C for a maximum profit of \$190.

2. Solving Minimum Problems in Standard Form Using the Duality Principle (25%)

Diet Preparation Mr. Jones needs to supplement his diet with at least 50 mg of calcium and 8 mg of iron daily. The minerals are available in two types of vitamin pills, P and Q. Pill P contains 5 mg of calcium and 2 mg of iron, while Pill Q contains 10 of mg calcium and 1 mg of iron. If each P pill costs 3 cents and each Q pill costs 4 cents, how could Mr. Jones minimize the cost of adding the minerals to his diet? What would the daily minimum cost be?

Let x_1 and x_2 represent the number of pill Ps and pill Qs, respectively, that Mr. Jones should add to his diet. He wishes to minimize his cost while meeting the nutritional requirements. He wants to

Minimize $C = 3x_1 + 4x_2$

subject to the conditions

$$\begin{aligned} 5x_1 + 10x_2 &\geq 50 \\ 2x_1 + x_2 &\geq 8 \\ x_1 \geq 0 \quad x_2 &\geq 0 \end{aligned}$$

The matrix representing the system and its transpose are

$$\text{matrix: } \left[\begin{array}{cc|c} 5 & 10 & 50 \\ 2 & 1 & 8 \\ 3 & 4 & 0 \end{array}\right] \qquad \text{transpose: } \left[\begin{array}{cc|c} 5 & 2 & 3 \\ 10 & 1 & 4 \\ 50 & 8 & 0 \end{array}\right]$$

The dual of the problem is

Maximize $P = 50y_1 + 8y_2$

subject to the conditions

$$\begin{aligned} 5y_1 + 2y_2 &\leq 3 \\ 10y_1 + y_2 &\leq 4 \\ y_1 \geq 0 \quad y_2 &\geq 0 \end{aligned}$$

We set up the initial simplex tableau and solve the maximum problem.

$$\begin{array}{c} \text{BV} \\ s_1 \\ \rightarrow s_2 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} P & y_1 & y_2 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 5 & 2 & 1 & 0 & 3 \\ 0 & \boxed{10} & 1 & 0 & 1 & 4 \\ \hline 1 & -50 & -8 & 0 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \text{BV} \\ \rightarrow s_1 \\ y_1 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} P & y_1 & y_2 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 0 & \boxed{\frac{3}{2}} & 1 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{10} & 0 & \frac{1}{10} & \frac{2}{5} \\ \hline 1 & 0 & -3 & 0 & 5 & 20 \end{array}\right] \end{array}$$

$$\longrightarrow \begin{array}{c} \text{BV} \\ y_2 \\ y_1 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} P & y_1 & y_2 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 0 & \boxed{1} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{15} & \frac{2}{15} & \frac{1}{3} \\ \hline 1 & 0 & 0 & 2 & 4 & 22 \end{array}\right] \end{array}$$

The solution to the minimum problem is minimal $C = 22$ when $x_1 = 2$ and $x_2 = 4$.
Mr. Jones can take his supplements with a minimum cost of \$0.22 per day if he takes 2 of pill P and 4 of pill Q.

3. The Simplex Method with Mixed Constraints (25%)

Shipping Schedule A television manufacturer must fill orders from two retailers. The first retailer, R1, has ordered 55 television sets, while the second retailer, R2, has ordered 75 sets. The manufacturer has the television sets stored in two warehouses, W1 and W2. There are 100 sets in W1 and 120 sets in W2. The shipping costs per television set are: \$8 from W1 to R1; \$12 from W1 to R2; \$13 from W2 to R1; \$7 from W2 to R2. Find the number of television sets to be shipped from each warehouse to each retailer if the total shipping cost is to be a minimum. What is this minimum cost?

We define x_1 as the number of television sets shipped from W_1 to R_1;

x_2 as the number shipped from W_1 to R_2;
x_3 as the number shipped from W_2 to R_1; and
x_4 as the number of sets shipped from W_2 to R_2.

The manufacturer wants to fill the orders at the lowest possible cost.

Minimize $\quad C = 8x_1 + 12x_2 + 13x_3 + 7x_4$

subject to the constraints

$$\begin{aligned} x_1 + x_3 &= 55 \\ x_2 + x_4 &= 75 \\ x_1 + x_2 &\le 100 \\ x_3 + x_4 &\le 120 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 &\ge 0,\ x_4 \ge 0 \end{aligned}$$

We rewrite the first two equations as inequalities, add nonnegative slack variables and set up the initial tableau.

$$\begin{aligned} x_1 + x_3 &\le 55 \\ -x_1 - x_3 &\le -55 \\ x_2 + x_4 &\le 75 \\ -x_2 - x_4 &\le -75 \\ x_1 + x_2 &\le 100 \\ x_3 + x_4 &\le 120 \\ x_1 \ge 0, x_2 \ge 0, x_3 &\ge 0,\ x_4 \ge 0 \end{aligned}$$

$$\begin{array}{c} \text{BV} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ P \end{array} \begin{array}{c} \begin{array}{ccccccccccc|c} P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \text{RHS} \end{array} \\ \left[\begin{array}{ccccccccccc|c} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 55 \\ 0 & \boxed{-1} & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -55 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -75 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 120 \\ \hline 1 & 8 & 12 & 13 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Since the initial tableau has negative entries in the RHS, we use the alternate pivoting strategy.

Alternative Pivoting Strategy $\rightarrow s_4$

$$
\begin{array}{c|ccccccccccc|c}
\text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \text{RHS} \\
\hline
s_1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 55 \\
s_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\
s_4 & 0 & 0 & \boxed{-1} & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -75 \\
s_5 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 45 \\
s_6 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 120 \\
\hline
P & 1 & 0 & 12 & 5 & 7 & 0 & 8 & 0 & 0 & 0 & 0 & -440
\end{array}
$$

Alternative Pivoting Strategy $\rightarrow x_2$

$$
\begin{array}{c|ccccccccccc|c}
\text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \text{RHS} \\
\hline
s_1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 55 \\
s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 75 \\
s_5 & 0 & 0 & 0 & \boxed{-1} & -1 & 0 & 1 & 0 & 1 & 1 & 0 & -30 \\
s_6 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 120 \\
\hline
P & 1 & 0 & 0 & 5 & -5 & 0 & 8 & 0 & 12 & 0 & 0 & -1{,}340
\end{array}
$$

Alternative Pivoting Strategy $\rightarrow x_2$

$$
\begin{array}{c|ccccccccccc|c}
\text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \text{RHS} \\
\hline
s_1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 25 \\
s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 75 \\
x_3 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & -1 & 0 & -1 & -1 & 0 & 30 \\
s_6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 90 \\
\hline
P & 1 & 0 & 0 & 0 & -10 & 0 & 13 & 0 & 17 & 5 & 0 & -1{,}490
\end{array}
$$

$$
\begin{array}{rcc}
 & \text{BV} & \begin{array}{ccccccccccccc} P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & & \text{RHS} \end{array} \\
\begin{array}{r} \\ \\ \\ \text{Standard Pivoting Strategy} \longrightarrow \\ \\ \\ \\ \end{array} &
\begin{array}{c} s_1 \\ x_1 \\ s_3 \\ x_2 \\ x_4 \\ s_6 \\ P \end{array} &
\left[\begin{array}{ccccccccccc|c}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 55 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & \boxed{1} & 0 & 45 \\
0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 90 \\
\hline
1 & 0 & 0 & 10 & 0 & 0 & 3 & 0 & 7 & -5 & 0 & -1{,}190
\end{array}\right]
\end{array}
$$

$$
\begin{array}{rcc}
 & \text{BV} & \begin{array}{ccccccccccccc} P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & & \text{RHS} \end{array} \\
\begin{array}{r} \\ \\ \\ \text{Standard Pivoting Strategy} \longrightarrow \\ \\ \\ \\ \end{array} &
\begin{array}{c} s_1 \\ x_1 \\ s_3 \\ s_5 \\ x_4 \\ s_6 \\ P \end{array} &
\left[\begin{array}{ccccccccccc|c}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 55 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 45 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 75 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 45 \\
\hline
1 & 0 & 5 & 5 & 0 & 0 & 8 & 0 & 7 & 0 & 0 & -965
\end{array}\right]
\end{array}
$$

The maximum $P = -965$, so the minimum $C = 965$ which is obtained when $x_1 = 55$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 75$. The television manufacturer minimizes shipping costs when 55 televisions are shipped from warehouse W_1 to retailer R_1, and 75 televisions from warehouse W_2 to retailer R_2 for a minimum cost of \$965.

4. Present Value of an Annuity; Amortization (25%)

House Mortgage A couple wish to purchase a house for \$200,000 with a down payment of \$40,000. They can amortize the balance either at 8% for 20 years or at 9% for 25 years. Which monthly payment is greater? For which loan is the total interest paid greater? After 10 years, which loan provides the greater equity?

If the couple chooses to finance their home with a 20 year mortgage at 8% interest, they will have $n = 12 \cdot 20 = 240$ monthly payments, the interest rate per payment period is $i = \frac{0.08}{12}$, and their monthly payments will be

$$P = V\frac{i}{1-(1+i)^{-n}} = \$160{,}000\,\frac{\frac{0.08}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-240}} = \$1338.30$$

The total interest paid on this loan is (\$1338.30)(240) – \$160,000 = \$161,192.

The couple's equity in the house is the sum of their down payment and the amount paid on the loan. After 10 years the couple still owes 120 payments. The amount still owed on the loan is the present value of these payments

$$V = P\frac{1-(1+i)^{-n}}{i} = \$1338.30\frac{1-\left(1+\frac{0.08}{12}\right)^{-120}}{\frac{0.08}{12}} = \$110{,}304.67$$

meaning the amount paid on the loan \$160,000 – \$110,304.67 = \$49,695.33.
The couple's equity after 10 years is \$40,000 + \$49,695.33 = \$89,695.33

If the couple chooses to finance their home with a 25 year mortgage at 9% interest, they will have $n = 12 \cdot 25 = 300$ monthly payments, the interest rate per payment period is $i = \frac{0.09}{12} = 0.0075$, and their monthly payments will be

$$P = V\frac{i}{1-(1+i)^{-n}} = \$160{,}000\,\frac{0.0075}{1-(1+0.0075)^{-300}} = \$1342.71$$

The total interest paid on this loan is (\$1342.71)(300) – \$160,000 = \$242,813.

The couple's equity in the house is the sum of their down payment and the amount paid on the loan. After 10 years the couple still owes 180 payments. The amount still owed on the loan is the present value of these payments

$$V = P\frac{1-(1+i)^{-n}}{i} = \$1342.71\frac{1-(1+0.0075)^{-180}}{0.0075} = \$132{,}382.36$$

meaning the amount paid on the loan \$160,000 – \$132,382.36 = \$27,617.64.
The couple's equity after 10 years is \$40,000 + \$27,617.64 = \$67,617.64.

The 25 year loan at 9% interest has a larger monthly payment, and the total interest paid on the 25 year loan is \$81,621 more than that on the 20 year loan.
After 10 years the couple would have more equity in their purchase if they had the 8% loan.