Test 2 for the Course of Management Mathematics

Date: June 12, 2008 Time: 9:30-11:30

1. Maximizing Profit (20%) (#43, P. 254)

A large TV manufacturer has warehouse facilities for storing its 52-inch color TVs in Chicago, New York, and Denver. Each month the city of Atlanta is shipped at most four hundred 52-inch TVs. The cost of transporting each TV to Atlanta from Chicago, New York, and Denver averages $20, $20, and $40, respectively, while the cost of labor required for packing averages $6, $8, and $4, respectively. Suppose $10,000 is allocated each month for transportation costs and $3000 is allocated for labor costs. If the profit on each TV made in Chicago is $50, in New York is $80, and in Denver is $40, how should monthly shipping arrangements be scheduled to maximize profit?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution

(a) Let P denote the profit, x_1 denote the number televisions shipped from Chicago, x_2 denote the number shipped from New York, and x_3 denote the number shipped from Denver.

Maximize

$$P = 70x_1 + 80x_2 + 40x_3$$

subject to the constraints

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &\le \quad 400 \\ 20x_1 + 20x_2 + 40x_3 &\le 10{,}000 \\ 6x_1 + \ 8x_2 + \ 4x_3 &\le \ 3{,}000 \\ x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 &\ge 0 \end{aligned}$$

Slack variables are added and the initial tableau is constructed and shown below.

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	RHS
s_1	0	[1]	1	1	1	0	0	400
s_2	0	20	20	40	0	1	0	10,000
s_3	0	6	8	4	0	0	1	3,000
P	1	−70	−80	−40	0	0	0	0

$R_2 = -20R_1 + r_2$
$R_3 = -6R_1 + r_3$
$R_4 = 70R_1 + r_4$

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	RHS
x_1	0	1	1	1	1	0	0	400
s_2	0	0	0	20	−20	1	0	2,000
s_3	0	0	[2]	−2	−6	0	1	600
P	1	0	−10	30	70	0	0	28,000

$R_3 = \frac{1}{2}r_3$
$R_1 = -R_3 + r_1$
$R_4 = 10R_3 + r_4$

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	RHS
x_1	0	1	0	2	4	0	$-\frac{1}{2}$	100
s_2	0	0	0	20	−20	1	0	2,000
x_2	0	0	1	−1	−3	0	$\frac{1}{2}$	300
P	1	0	0	20	40	0	5	31,000

Since all the entries in the objective row are nonnegative, this is the final tableau. The maximum $P = 31{,}000$, obtained when $x_1 = 100$, $x_2 = 300$, and $x_3 = 0$.

(c) The manufacturer will obtain a maximum profit of \$31,000, if it ships 100 televisions from Chicago, 300 televisions from New York and none from Denver.

2. Production Schedule (20%) (#30, P. 266)

A company owns two mines. Mine A produces 1 ton of high-grade ore, 3 tons of medium-grade ore, and 5 tons of low-grade ore each day.Mine B produces 2 tons of each grade ore per day. The company needs at least 80 tons of high-grade ore, at least 160 tons of medium-grade ore, and at least 200 tons of low-grade ore. How many days should each mine be operated to minimize costs if it costs $2000 per day to operate each mine?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) nalyze the solution

(a) Let C represent the daily cost of operating the two mines, and let x_1 and x_2 represent the number of days mine A and mine B, respectively are operated. The company wants to minimize the cost of operation while meeting demand.

Minimize

$$C = 2000x_1 + 2000x_2$$

subject to the constraints

$$\begin{aligned} x_1 + 2x_2 &\geq 80 \\ 3x_1 + 2x_2 &\geq 160 \\ 5x_1 + 2x_2 &\geq 200 \\ x_1 \geq 0 \quad x_2 &\geq 0 \end{aligned}$$

(b) The matrix (on the left) representing the system and its transpose (on the right) are:

$$\begin{array}{cc} x_1 & x_2 \end{array} \qquad \qquad \begin{array}{ccc} y_1 & y_2 & y_3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 80 \\ 3 & 2 & 160 \\ 5 & 2 & 200 \\ 2000 & 2000 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 2000 \\ 2 & 2 & 2 & 2000 \\ 80 & 160 & 200 & 0 \end{array}\right]$$

The dual of the problem is

Maximize

$$P = 80y_1 + 160y_2 + 200y_3$$

subject to the constraints

$$\begin{aligned} y_1 + 3y_2 + 5y_3 &\leq 2000 \\ 2y_1 + 2y_2 + 2y_3 &\leq 2000 \\ y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 &\geq 0 \end{aligned}$$

We set up the initial simplex tableau and solve the maximum problem.

$$\begin{array}{c} \text{BV} \\ s_1 \\ s_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccccc} P & y_1 & y_2 & y_3 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 3 & 5 & 1 & 0 & 2000 \\ 0 & \boxed{2} & 2 & 2 & 0 & 1 & 2000 \\ \hline 1 & -80 & -160 & -200 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\xrightarrow[\substack{R_2 = \frac{1}{2}r_2 \\ R_1 = -R_2 + r_1 \\ R_3 = 80R_2 + r_3}]{} \begin{array}{c} \text{BV} \\ s_1 \\ y_1 \\ P \end{array} \begin{array}{c} \begin{array}{ccccccc} P & y_1 & y_2 & y_3 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & \boxed{2} & 4 & 1 & -\frac{1}{2} & 1{,}000 \\ 0 & 1 & 1 & 1 & 0 & \frac{1}{2} & 1{,}000 \\ \hline 1 & 0 & -80 & -120 & 0 & 40 & 80{,}000 \end{array}\right] \end{array}$$

$$\xrightarrow[\substack{R_1 = \frac{1}{2}r_1 \\ R_2 = -R_1 + r_2 \\ R_3 = 80R_1 + r_3}]{} \begin{array}{c} \text{BV} \\ y_2 \\ y_1 \\ P \end{array} \begin{array}{c} \begin{array}{ccccccc} P & y_1 & y_2 & y_3 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 2 & \frac{1}{2} & -\frac{1}{4} & 500 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{3}{4} & 500 \\ \hline 1 & 0 & 0 & 40 & 40 & 20 & 120{,}000 \end{array}\right] \end{array}$$

Minimal $C = 120{,}000$ when $x_1 = 40$ and $x_2 = 20$.

(c) The mining company can minimize its costs while meeting demand if it operates mine A for 40 days and mine B for 20 days. The minimum cost is \$120,000.

3. **Shipping Schedule (20%) (#17, P. 281)**

A motorcycle manufacturer must fill orders from two dealers. The first dealer, D1, has ordered 20 motorcycles, while the second dealer, D2, has ordered 30 motorcycles. The manufacturer has the motorcycles stored in two warehouses,W1 and W2. There are 40 motorcycles in W1 and 15 in W2. The shipping costs per motorcycle are as follows: $15 from W1 to D1; $13 from W1 to D2; $14 from W2 to D1; $16 from W2 to D2. Under these conditions, find the number of motorcycles to be shipped from each warehouse to each dealer if the total shipping cost is to be held to a minimum. What is this minimum cost?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) analyze the solution

(a) Let C denote the total shipping cost, and x_1 the number of motorcycles shipped from W_1 to D_1; x_2 the number shipped from W_1 to D_2; x_3 the number shipped from W_2 to D_1; and x_4 the number of motorcycles shipped from W_2 to D_2.

The manufacturer wants to fill the orders at the lowest possible cost.

Minimize

$$C = 15x_1 + 13x_2 + 14x_3 + 16x_4$$

subject to the constraints

$$\begin{aligned} x_1 + x_3 &= 20 \\ x_2 + x_4 &= 30 \\ x_1 + x_2 &\le 40 \\ x_3 + x_4 &\le 15 \end{aligned}$$

$$x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0 \quad x_4 \ge 0$$

(b) This is a minimum problem that is not in standard form. We will solve the maximum problem formed by writing $P = -C = -15x_1 - 13x_2 - 14x_3 - 16x_4$. Add nonnegative slack variables and set up the initial tableau. Since there are equality constraints, we first pivot on the nonbasic variables.

BV	P	x_1	x_2	x_3	x_4	s_1	s_2	RHS
	0	1	0	1	0	0	0	20
	0	0	1	0	1	0	0	30
s_1	0	1	1	0	0	1	0	40
s_2	0	0	0	1	1	0	1	15
P	1	15	13	14	16	0	0	0

Nonbasic Variable Pivoting Strategy →

BV	P	x_1	x_2	x_3	x_4	s_1	s_2	RHS
x_1	0	1	0	1	0	0	0	20
	0	0	1	0	1	0	0	30
s_1	0	0	1	−1	0	1	0	20
s_2	0	0	0	1	1	0	1	15
P	1	0	13	−1	16	0	0	−300

Nonbasic Variable Pivoting Strategy →

BV	P	x_1	x_2	x_3	x_4	s_1	s_2	RHS
x_1	0	1	0	1	0	0	0	20
x_2	0	0	1	0	1	0	0	30
s_1	0	0	0	−1	−1	1	0	−10
s_2	0	0	0	1	1	0	1	15
P	1	0	0	−1	3	0	0	−690

Since the initial tableau has negative entries in the RHS, we use the Alternative Pivoting Strategy.

Alternative Pivoting Strategy →

BV	P	x_1	x_2	x_3	x_4	s_1	s_2	RHS
x_1	0	1	0	0	−1	1	0	10
x_2	0	0	1	0	1	0	0	30
x_3	0	0	0	1	1	−1	0	10
s_2	0	0	0	0	0	1	1	5
P	1	0	0	0	4	−1	0	−680

We now use the Standard Pivoting Strategy since there are no negative entries in the RHS.

Standard Pivoting Strategy →

BV	P	x_1	x_2	x_3	x_4	s_1	s_2	RHS
x_1	0	1	0	0	−1	0	−1	5
x_2	0	0	1	0	1	0	0	30
x_3	0	0	0	1	1	0	1	15
s_1	0	0	0	0	0	1	1	5
P	1	0	0	0	4	0	1	−675

Minimum $C = 675$ when $x_1 = 5$, $x_2 = 30$, $x_3 = 15$, $x_4 = 0$.

(c) The motorcycle manufacturer minimizes shipping costs at $675 by shipping 5 motorcycles from warehouse W_1 to dealer D_1, shipping 30 motorcycles from W_1 to D_2, and 15 motorcycles from W_2 to D_1.

4. Finding the Time Required to Double/Triple an Investment (20%)

(a) How long will it take for an investment to double in value if it earns 5% compounded monthly?

(b) How long will it take to triple at this rate?

SOLUTION **(a)** If P is the initial investment and we want P to double, the amount A will be $2P$. We use the Compound Interest Formula (1) with $i = \frac{0.05}{12}$. Then

$$A = P(1 + i)^n \quad \text{Formula (1)}$$

$$2P = P\left(1 + \frac{0.05}{12}\right)^n \quad A = 2P,\ i = \frac{0.05}{12}$$

$$2 = (1.0041667)^n \quad \text{Simplify.}$$

$$n = \log_{1.0041667} 2 = \frac{\log_{10} 2}{\log_{10} 1.0041667} = 166.7 \text{ months}$$

↑ Apply the definition of a logarithm. ↑ Change-of-base formula ↑ Payment period measured in months

It will take about 13 years 11 months to double the investment.

(b) To triple the investment, the amount A is $3P$. Thus,

$$A = P(1 + i)^n \quad \text{Formula (1)}$$

$$3P = P\left(1 + \frac{0.05}{12}\right)^n \quad A = 3P,\ i = \frac{0.05}{12}$$

$$3 = (1.0041667)^n \quad \text{Simplify.}$$

$$n = \log_{1.0041667} 3 = \frac{\log_{10} 3}{\log_{10} 1.0041667} = 264.2 \text{ months}$$

↑ Apply the definition of a logarithm

It will take about 22 years to triple the investment.

5. Lease or Purchase (20%) (#4, P.330)

A corporation may obtain a particular machine either by leasing it for 4 years (the useful life) at an annual rent of $1000 or by purchasing the machine for $3000.

(a) Which alternative is preferable if the corporation can invest money at 10% per annum?

(b) What if it can invest at 14% per annum?

Solution

(a) Suppose the corporation may invest money at 10% per annum. The present value of an annuity of $1000 for 4 years at 10% equals $3169.87, which exceeds the purchase price. Therefore, purchase is preferable.

(b) Suppose the corporation may invest at 14% per annum. The present value of an annuity of $1000 for 4 years at 14% equals $2913.71, which is less than the purchase price. Leasing is preferable.