Test 1 for the Course of Management Mathematics

Date: April 10, 2008   Time: 9:30-11:30

Mixture Problem (25%)

A coffee manufacturer wants to market a new blend of coffee that will cost $6.00 per pound by mixing $5.00 per pound coffee and $7.50 per pound coffee. What amounts of the $5.00 per pound coffee and $7.50 per pound coffee should be blended to obtain the desired mixture?

[*Hint:* Assume the total weight of the desired blend is 100 pounds.]

. We will use a table to organize the information, and we will use the hint and assume that 100 pounds of blended coffee will be made.

Coffees	Pounds of Coffee	Price per Pound	Total Cost of Coffee
Type 1	x	$5.00	$5.00x$
Type 2	$y = 100 - x$	$7.50	$7.50y = 7.50(100 - x)$
Blend	$x + y = 100$	$6.00	$6.00(100) = 600$

The last column provides the information necessary to solve the problem, since the sum of the costs of the individual coffees must equal the cost of the blend.

$$\begin{aligned} 5x + 7.5(100 - x) &= 600 \\ 5x + 750 - 7.5x &= 600 \\ -2.5x &= -150 \\ x &= 60 \end{aligned}$$

The manufacturer should mix 60 pounds of type 1 coffee and 40 pounds of type 2 coffee to get 100 pounds of a blend worth $6.00 per pound.

Maximizing Profit (25%)

A company makes two explosives: type I and type II. Due to storage problems, a maximum of 100 pounds of type I and 150 pounds of type II can be mixed and packaged each week. One pound of type I takes 60 hours to mix and 70 hours to package; 1 pound of type II takes 40 hours to mix and 40 hours to package. The mixing department has at most 7200 work-hours available each week, and packaging has at most 7800 work-hours available. If the profit for 1 pound of type I is \$60 and for 1 pound of type II is \$40, what is the maximum profit possible each week?

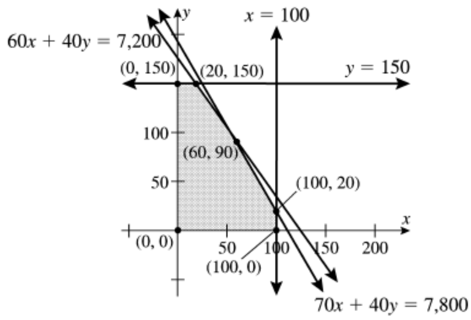
Let x = pounds of type I explosive made, and
y = pounds of type II explosive made.

The objective is to maximize the profit, P, from selling the explosives. P is given by:

$P = 60x + 40y$

The storage problems and the work-hours available form constraints on the problem:

$$\begin{cases} x \le 100 & \text{storage limit} & (1) \\ y \le 150 & \text{storage limit} & (2) \\ 60x + 40y \le 7200 & \text{mixing hours available} & (3) \\ 70x + 40y \le 7800 & \text{packaging hours available} & (4) \\ x \ge 0 & & (5) \\ y \ge 0 & & (6) \end{cases}$$



The constraints are graphed and the set of feasible points is shaded. The corner points are (0, 0), (100, 0), (0, 150), (20, 150), (100, 20), and (60, 90). The point (60, 90) is found by solving the system of equations $\begin{cases} 60x + 40y = 7200 & (1) \\ 70x + 40y = 7800 & (2) \end{cases}$. When (1) is subtracted from (2), we get $10x = 600$ or $x = 60$. Back-substituting into (1) gives $40y = 3600$ or $y = 90$.

The objective function P is evaluated at the corner points as shown in the table.

Corner Point (x, y)	Value of Objective Function $P = 60x + 40y$
(0, 0)	$P = 60(0) + 40(0) = 0$
(100, 0)	$P = 60(100) + 40(0) = 6000$
(0, 150)	$P = 60(0) + 40(150) = 6000$
(20, 150)	$P = 60(20) + 40(150) = 7200$
(100, 20)	$P = 60(100) + 40(20) = 6800$
(60, 90)	$P = 60(60) + 40(90) = 7200$

The maximum profit is \$7200. It is attained when 20 pounds of type I and 150 pounds of type II explosive are made, when 60 pounds of type I and 90 pounds of type II explosive are made or when a combination of the explosives, satisfying the equation $60x + 40y = 7200$, $20 \le x \le 60$ is made.

Open Leontief System (25%)

Suppose three corporations each manufacture a different product, although each needs the others' goods to produce its own. Moreover, suppose some of each manufacture's production is consumed by individuals outside the system. Current internal and external consumptions are give in the table below, but it has been forecast that consumer demand will change in the next several years and that in 5 years consumers will demand 80 units of the product A, 40 units of product B, and 80 units of product C. Find the total output needed to be made by each corporation to meet the predicted demand.

Amounts Consumed

	A	B	C	Consumer	Total
A	100	50	40	60	250
B	20	10	30	40	100
C	30	40	30	100	200

Amount Produced

- This is an open Leontief system. To find the total output to be produced by each corporation to meet predicted future demand, we need to define matrices A and D_5 and to solve the equation $X = [I_3 - A]^{-1}D_5$.

$$A = \begin{bmatrix} \frac{100}{250} & \frac{50}{100} & \frac{40}{200} \\ \frac{20}{250} & \frac{10}{100} & \frac{30}{200} \\ \frac{30}{250} & \frac{40}{100} & \frac{30}{200} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 & 0.2 \\ 0.08 & 0.1 & 0.15 \\ 0.12 & 0.4 & 0.15 \end{bmatrix}; D_5 = \begin{bmatrix} 80 \\ 40 \\ 80 \end{bmatrix}$$

$$X = [I_3 - A]^{-1}D_5 = \begin{bmatrix} 0.6 & -0.5 & -0.2 \\ -0.08 & 0.9 & -0.15 \\ -0.12 & -0.4 & 0.85 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 40 \\ 80 \end{bmatrix} = \begin{bmatrix} 275.57 \\ 98.86 \\ 179.55 \end{bmatrix}$$

To meet the predicted future demand 275.57 units of A, 98.86 units of B, and 179.55 units of C should be produced.

The Inverse of a Matrix (25%)

Show that the inverse of $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the formula

$A^{-1}=\begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$ where $\Delta = ad - bc \neq 0$.(The number Δ is called the determinant of A)

, To show that the inverse of A is given by $\begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$ where $\Delta = ad - bc \neq 0$ we need to show that $AA^{-1} = A^{-1}A = I_2$.

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} = \begin{bmatrix} \frac{ad}{\Delta}-\frac{bc}{\Delta} & -\frac{ab}{\Delta}+\frac{ab}{\Delta} \\ \frac{cd}{\Delta}-\frac{cd}{\Delta} & -\frac{bc}{\Delta}+\frac{ad}{\Delta} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{\Delta} & 0 \\ 0 & \frac{ad-bc}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^{-1}A = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{ad}{\Delta}-\frac{bc}{\Delta} & \frac{bd}{\Delta}-\frac{bd}{\Delta} \\ -\frac{ac}{\Delta}+\frac{ac}{\Delta} & -\frac{bc}{\Delta}+\frac{ad}{\Delta} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{\Delta} & 0 \\ 0 & \frac{ad-bc}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

So, $\begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.