mathematical modeling. The material in this example will be returned to in subsequent sections so that we can analyze it in greater detail and from different points of view.

EXAMPLE 7 **Price–Demand and Revenue Modeling** A manufacturer of a popular automatic camera wholesales the camera to retail outlets throughout the United States. Using statistical methods, the financial department in the company produced the price–demand data in Table 4, where p is the wholesale price per camera at which x million cameras are sold. Notice that as the price goes down, the number sold goes up.

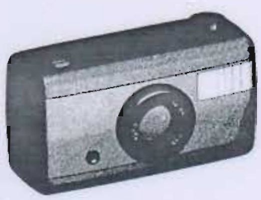


TABLE 4 Price–Demand

x (Millions)	p ($)
2	87
5	68
8	53
12	37

TABLE 5 Revenue

x (Millions)	$R(x)$ (Million $)
1	90
3	
6	
9	
12	
15	

Using special analytical techniques (regression analysis), an analyst arrived at the following price–demand function that models the Table 4 data:

$$p(x) = 94.8 - 5x \qquad 1 \le x \le 15 \qquad (5)$$

(A) Plot the data in Table 4. Then sketch a graph of the price–demand function in the same coordinate system.

(B) What is the company's revenue function for this camera, and what is the domain of this function?

(C) Complete Table 5, computing revenues to the nearest million dollars.

(D) Plot the data in Table 5. Then sketch a graph of the revenue function using these points.

(E) Plot the revenue function on a graphing utility.

Solution (A)

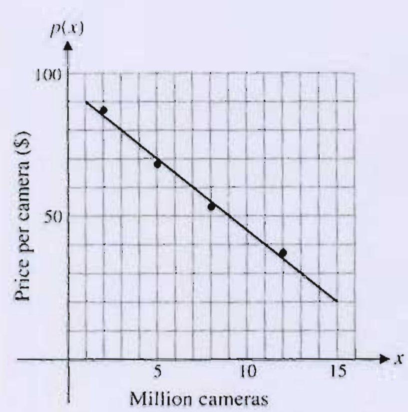


FIGURE 7 Price–demand

In Figure 7, notice that the model approximates the actual data in Table 4, and it is assumed that it gives realistic and useful results for all other values of x between 1 million and 15 million.

(B) $R(x) = xp(x) = x(94.8 - 5x)$ million dollars

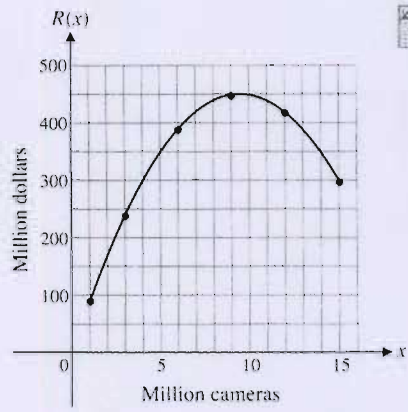
Domain: $1 \le x \le 15$

[Same domain as the price–demand function, equation (5).]

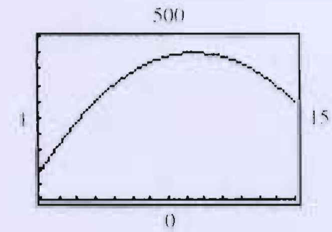
(C) **TABLE 5 Revenue**

x (Millions)	$R(x)$ (Million \$)
1	90
3	239
6	389
9	448
12	418
15	297

(D)



(E)



Matched Problem 7

The financial department in Example 7, using statistical techniques, produced the data in Table 6, where $C(x)$ is the cost in millions of dollars for manufacturing and selling x million cameras.

Using special analytical techniques (regression analysis), an analyst produced the following cost function to model the data:

$$C(x) = 156 + 19.7x \qquad 1 \le x \le 15 \tag{6}$$

TABLE 6 Cost Data

x (Millions)	$C(x)$ (Million \$)
1	175
5	260
8	305
12	395

(A) Plot the data in Table 6. Then sketch a graph of equation (6) in the same coordinate system.

(B) What is the company's profit function for this camera, and what is its domain?

(C) Complete Table 7, computing profits to the nearest million dollars.

TABLE 7 Profit

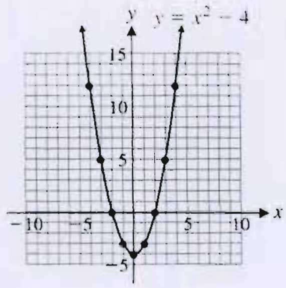
x (Millions)	$P(x)$ (Million \$)
1	−86
3	
6	
9	
12	
15	

(D) Plot the points from part (C). Then sketch a graph of the profit function through these points.

(E) Plot the profit function on a graphing utility.

Answers to Matched Problems

1.



2. (A) Does not specify a function
(B) Specifies a function

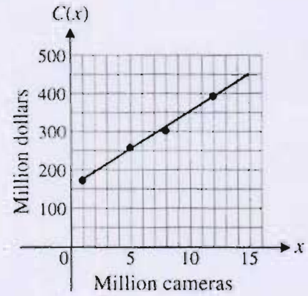
3. $x \ge 2$ (inequality notation) or $[2, \infty)$ (interval notation)

4. (A) -3 (B) 0
(C) Does not exist (D) 6

5. Domain of F: R; domain of G: all real numbers except -3;
domain of H: $x \le 2$ (inequality notation) or $(-\infty, 2]$ (interval notation)

6. (A) $a^2 - 4a + 9$ (B) $a^2 + 2ah + h^2 - 4a - 4h + 9$
(C) $2ah + h^2 - 4h$ (D) $2a + h - 4$

7. (A)

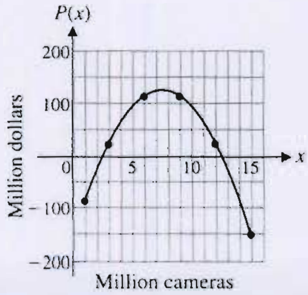


(B) $P(x) = R(x) - C(x) = x(94.8 - 5x) - (156 + 19.7x)$; domain: $1 \le x \le 15$

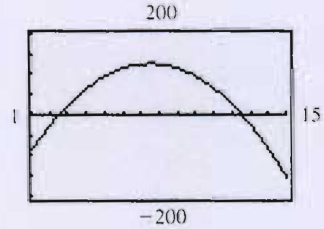
(C) **TABLE 7** Profit

x (Millions)	$P(x)$ (Million \$)
1	−86
3	24
6	115
9	115
12	25
15	−155

(D)



(E)



Step 2. *Solve* the mathematical model.

Step 3. *Interpret* the solution to the mathematical model in terms of the original real-world problem.

In more complex problems, this cycle may have to be repeated several times to obtain the required information about the real-world problem.

Example 6 **Purchasing** A company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic feet. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many of each type of truck should the company purchase?

Solution The question in this example indicates that the relevent variables are the number of each type of truck:

$$x_1 = \text{number of 10-foot trucks}$$
$$x_2 = \text{number of 14-foot trucks}$$
$$x_3 = \text{number of 24-foot trucks}$$

Next we form the mathematical model:

$$\begin{aligned} x_1 + x_2 + x_3 &= 25 \quad \text{Total number of trucks} \\ 350x_1 + 700x_2 + 1{,}400x_3 &= 28{,}000 \quad \text{Total capacity} \end{aligned} \qquad (2)$$

Now we form the augmented coefficient matrix of the system and solve by using Gauss–Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 350 & 700 & 1{,}400 & 28{,}000 \end{array}\right] \quad \tfrac{1}{350}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 1 & 2 & 4 & 80 \end{array}\right] \quad -R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad -R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -30 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad \text{Matrix is in reduced form.}$$

$$x_1 - 2x_3 = -30 \quad \text{or} \quad x_1 = 2x_3 - 30$$
$$x_2 + 3x_3 = 55 \quad \text{or} \quad x_2 = -3x_3 + 55$$

Let $x_3 = t$. Then for t any real number,

$$\begin{aligned} x_1 &= 2t - 30 \\ x_2 &= -3t + 55 \\ x_3 &= t \end{aligned} \qquad (3)$$

is a solution to mathematical model (2).

Now we must interpret this solution in terms of the original problem. Since the variables x_1, x_2, and x_3 represent numbers of trucks, they must be nonnegative real numbers. And since we can't purchase a fractional number of trucks, each must be a nonnegative whole number. Since $t = x_3$, it follows that t must also be a nonnegative whole number. The first and second equations in model (3) place additional restrictions on the values that t can assume:

$$x_1 = 2t - 30 \geq 0 \quad \text{implies that} \quad t \geq 15$$

$$x_2 = -3t + 55 \geq 0 \quad \text{implies that} \quad t \leq \frac{55}{3} = 18\tfrac{1}{3}$$

Thus, the only possible values of t that will produce meaningful solutions to the original problem are 15, 16, 17, and 18. That is, the only combinations of 25 trucks that will result in a combined capacity of 28,000 cubic feet are $x_1 = 2t - 30$ 10-foot trucks, $x_2 = -3t + 55$ 14-foot trucks, and $x_3 = t$ 24-foot trucks, where $t = 15, 16, 17$, or 18. A table is a convenient way to display these solutions:

	10-Foot Truck	14-Foot Truck	24-Foot Truck
t	x_1	x_2	x_3
15	0	10	15
16	2	7	16
17	4	4	17
18	6	1	18

Matched Problem 6 A company that rents small moving trucks wants to purchase 16 trucks with a combined capacity of 19,200 cubic feet. Three different types of trucks are available: a cargo van with a capacity of 300 cubic feet, a 15-foot truck with a capacity of 900 cubic feet, and a 24-foot truck with a capacity of 1,500 cubic feet. How many of each type of truck should the company purchase?

EXPLORE & DISCUSS 3 Refer to Example 6. The rental company charges \$19.95 per day for a 10-foot truck, \$29.95 per day for a 14-foot truck, and \$39.95 per day for a 24-foot truck. Which of the four possible choices in the table would produce the largest daily income from truck rentals?

Answers to Matched Problems

1. (A) Condition 2 is violated: The 3 in row 2 and column 2 should be a 1. Perform the operation $\frac{1}{3}R_2 \rightarrow R_2$ to obtain

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$$

(B) Condition 3 is violated: The 5 in row 1 and column 2 should be a 0. Perform the operation $(-5)R_2 + R_1 \rightarrow R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 8 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C) Condition 4 is violated. The leftmost 1 in the second row is not to the right of the leftmost 1 in the first row. Perform the operation $R_1 \leftrightarrow R_2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(D) Condition 1 is violated: The all-zero second row should be at the bottom. Perform the operation $R_2 \leftrightarrow R_3$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2. $x_1 = 1, x_2 = -1, x_3 = 0$
3. Inconsistent; no solution
4. $x_1 = 5t + 4, x_2 = 3t + 5, x_3 = t$, t any real number
5. $x_1 = s + 7, x_2 = s, x_3 = t - 2, x_4 = -3t - 1, x_5 = t$, s and t any real numbers
6. $t - 8$ cargo vans, $-2t + 24$ 15-foot trucks, and t 24-foot trucks, where $t = 8, 9, 10, 11$, or 12

Exercise 4-3

A *In Problems 1–10, if a matrix is in reduced form, say so. If not, explain why and indicate the row operation(s) necessary to transform the matrix into reduced form.*

1. $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$

2. $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & -1 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

7. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

9. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

10. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

Write the linear system corresponding to each reduced augmented matrix in Problems 11–18 and solve.

11. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

12. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

15. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

16. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$

17. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -7 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 6 \end{array}\right]$

19. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -5 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$

20. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 4 \\ 0 & 1 & -1 & 2 & -1 \end{array}\right]$

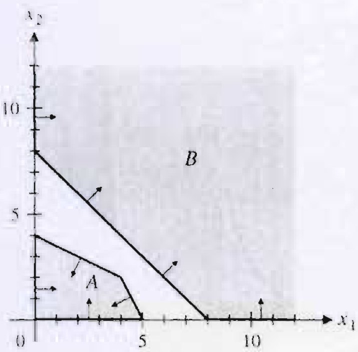
EXPLORE & DISCUSS 2

In Example 2B we saw that there was no optimal solution for the problem of maximizing the objective function P over the feasible region S. We want to add an additional constraint to modify the feasible region so that an optimal solution for the maximization problem does exist. Which of the following constraints will accomplish this objective?

(A) $x_1 \le 20$ (B) $x_2 \ge 4$ (C) $x_1 \le x_2$ (D) $x_2 \le x_1$

For an illustration of Theorem 2C, consider the following:

$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \ge 8 \\ & x_1 + 2x_2 \le 8 \\ & 2x_1 + x_2 \le 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$



$A \cap B = \varnothing$

FIGURE 3

The intersection of the graphs of the constraint inequalities is the empty set (Fig. 3); hence, the *feasible region is empty* (see Section 6-2). If this happens, the problem should be reexamined to see if it has been formulated properly. If it has, the management may have to reconsider items such as labor-hours, overtime, budget, and supplies allocated to the project in order to obtain a nonempty feasible region and a solution to the original problem.

➤ Applications

EXAMPLE 3

Medicine We now convert Example 5 from the preceding section into a linear programming problem. A patient in a hospital is required to have at least 84 units of drug A and 120 units of drug B, each day (assume that an overdosage of either drug is harmless). Each gram of substance M contains 10 units of drug A and 8 units of drug B, and each gram of substance N contains 2 units of drug A and 4 units of drug B. Now, suppose that both M and N contain an undesirable drug D, 3 units per gram in M and 1 unit per gram in N. How many grams of each of substances M and N should be mixed to meet the minimum daily requirements and at the same time minimize the intake of drug D? How many units of the undesirable drug D will be in this mixture?

Solution First we construct the mathematical model.

Step 1. Introduce decision variables. According to the questions asked in this example, we must decide how many grams of substances M and N should be mixed to form the daily dose of medication. Thus, these two quantities are the decision variables:

$$x_1 = \text{number of grams of substance } M \text{ used}$$
$$x_2 = \text{number of grams of substance } N \text{ used}$$

Step 2. Summarize relevant material in a table, relating the columns to substances M and N.

	Amount of Drug per Gram		Minimum Daily
	Substance *M*	**Substance *N***	**Requirement**
Drug *A*	10 units	2 units	84 units
Drug *B*	8 units	4 units	120 units
Drug *D*	3 units	1 unit	

Step 3. Determine the objective and the objective function. The objective is to minimize the amount of drug D in the daily dose of medication. Using the decision variables and the information in the table, we form the linear objective function

$$C = 3x_1 + x_2$$

Step 4. Write the problem constraints. The constraints in this problem involve minimum requirements, so the inequalities will take a different form:

$$\begin{aligned} 10x_1 + 2x_2 &\geq 84 && \text{Drug A constraint} \\ 8x_1 + 4x_2 &\geq 120 && \text{Drug B constraint} \end{aligned}$$

Step 5. Add the nonnegative constraints and summarize the model.

$$\begin{aligned} \text{Minimize} \quad & C = 3x_1 + x_2 && \text{Objective function} \\ \text{subject to} \quad & 10x_1 + 2x_2 \geq 84 && \text{Drug A constraint} \\ & 8x_1 + 4x_2 \geq 120 && \text{Drug B constraint} \\ & x_1, x_2 \geq 0 && \text{Nonnegative constraints} \end{aligned}$$

Now we use the geometric method to solve the problem.

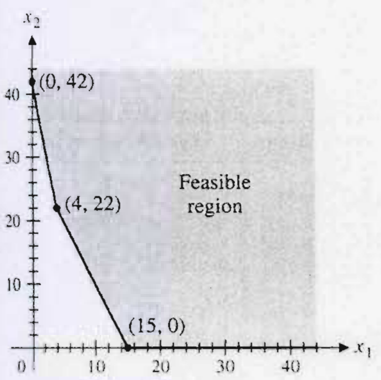


FIGURE 4

Step 1. Graph the feasible region. Then, after checking Theorem 2 to determine that an optimal solution exists, find the coordinates of each corner point. Solving the system of constraint inequalities graphically, we obtain the feasible region shown in Figure 4. Since the feasible region is unbounded and the coefficients of the objective function are positive, this minimization problem has a solution.

Step 2. Evaluate the objective function at each corner point, as shown in the table.

Corner Point (x_1, x_2)	$C = 3x_1 + x_2$
(0, 42)	42
(4, 22)	34
(15, 0)	45

Step 3. Determine the optimal solution from step 2. The optimal solution is $C = 34$ at the corner point (4, 22).

Step 4. Interpret the optimal solution in terms of the original problem. If we use 4 grams of substance M and 22 grams of substance N, we will supply the minimum daily requirements for drugs A and B and minimize the intake of the undesirable drug D at 34 units. (Any other combination of M and N from the feasible region will result in a larger amount of the undesirable drug D.) ■

Matched Problem 3

Agriculture A chicken farmer can buy a special food mix A at 20¢ per pound and a special food mix B at 40¢ per pound. Each pound of mix A contains 3,000 units of nutrient N_1 and 1,000 units of nutrient N_2; each pound of mix B contains 4,000 units of nutrient N_1 and 4,000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36,000 units of nutrient N_1 and 20,000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? Construct a mathematical model and solve using the geometric method.

Insight Refer to Example 3. If we change the minimum requirement for drug A from 120 to 125, the optimal solution changes to 3.6 grams of substance M and 24.1 grams of substance N, correct to one decimal place.

Now refer to Example 1. If we change the maximum labor-hours available per day in the assembly department from 84 to 79, the solution changes to 15 standard tents and 8.5 expedition tents.

We can measure 3.6 grams of substance M and 24.1 grams of substance N, but how can we make 8.5 tents? Should we make 8 tents? Or 9 tents? If the solutions to a problem must be integers and the optimal solution found graphically involves decimals, rounding the decimal value to the nearest integer does not always produce the *optimal integer solution* (see Problem 36, Exercise 5-2). Finding optimal integer solutions to a linear programming problem is called *integer programming* and requires special techniques that are beyond the scope of this book. As mentioned earlier, if we encounter a solution like 8.5 tents per day, we will interpret this as an *average* value over many days of production. ●

Answers to Matched Problems

1. (A) x_1 = number of two-person boats produced each month
x_2 = number of four-person boats produced each month

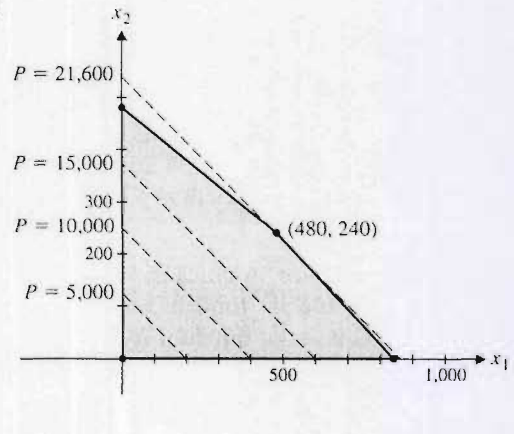
(B)

	Labor-Hours Required		Maximum Labor-Hours Available per Month
	Two-Person Boat	**Four-Person Boat**	
Cutting department	0.9	1.8	864
Assembly department	0.8	1.2	672
Profit per boat	\$25	\$40	

(C) $P = 25x_1 + 40x_2$

(D) $0.9x_1 + 1.8x_2 \le 864$
$0.8x_1 + 1.2x_2 \le 672$
$x_1, x_2 \ge 0$

(E)



(F) 480 two-person boats, 240 four-person boats; max P = \$21,600 per month

2. (A) Min $z = 24$ at $(3, 6)$; max $z = 40$ at $(2, 16)$ and $(8, 4)$ (multiple optimal solution)

(B) Min $z = 90$ at $(0, 18)$; no maximum value

3. Min $C = 0.2x_1 + 0.4x_2$
subject to $3{,}000x_1 + 4{,}000x_2 \ge 36{,}000$
$1{,}000x_1 + 4{,}000x_2 \ge 20{,}000$
$x_1, x_2 \ge 0$

8 lb of mix A, 3 lb of mix B; min C = \$2.80 per day