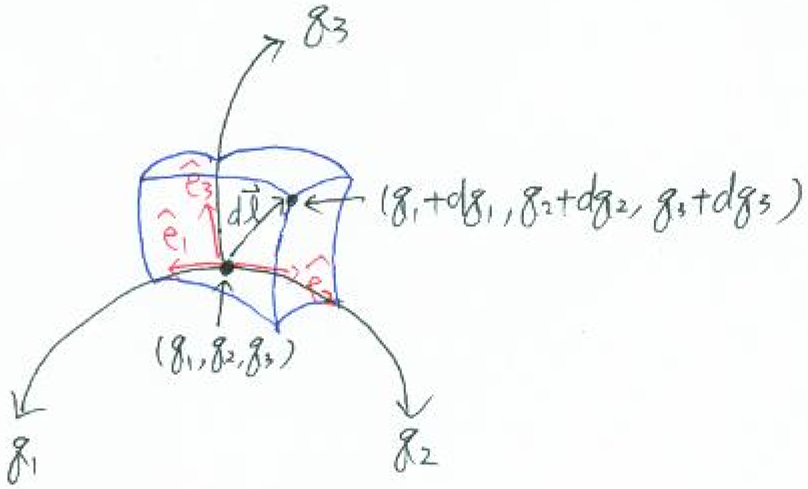
座標	對稱性
直角	無
柱	軸
球	心

§ 曲線座標(任意座標)



座標	q_1	q_2	q_3
基底	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$
尺度因子	h_1	h_2	h_3

$$d\vec{l} = \hat{e}_1 h_1 dq_1 + \hat{e}_2 h_2 dq_2 + \hat{e}_3 h_3 dq_3$$

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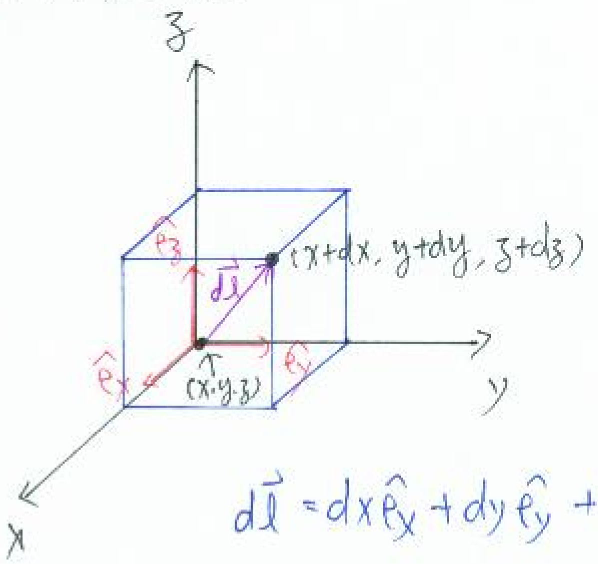
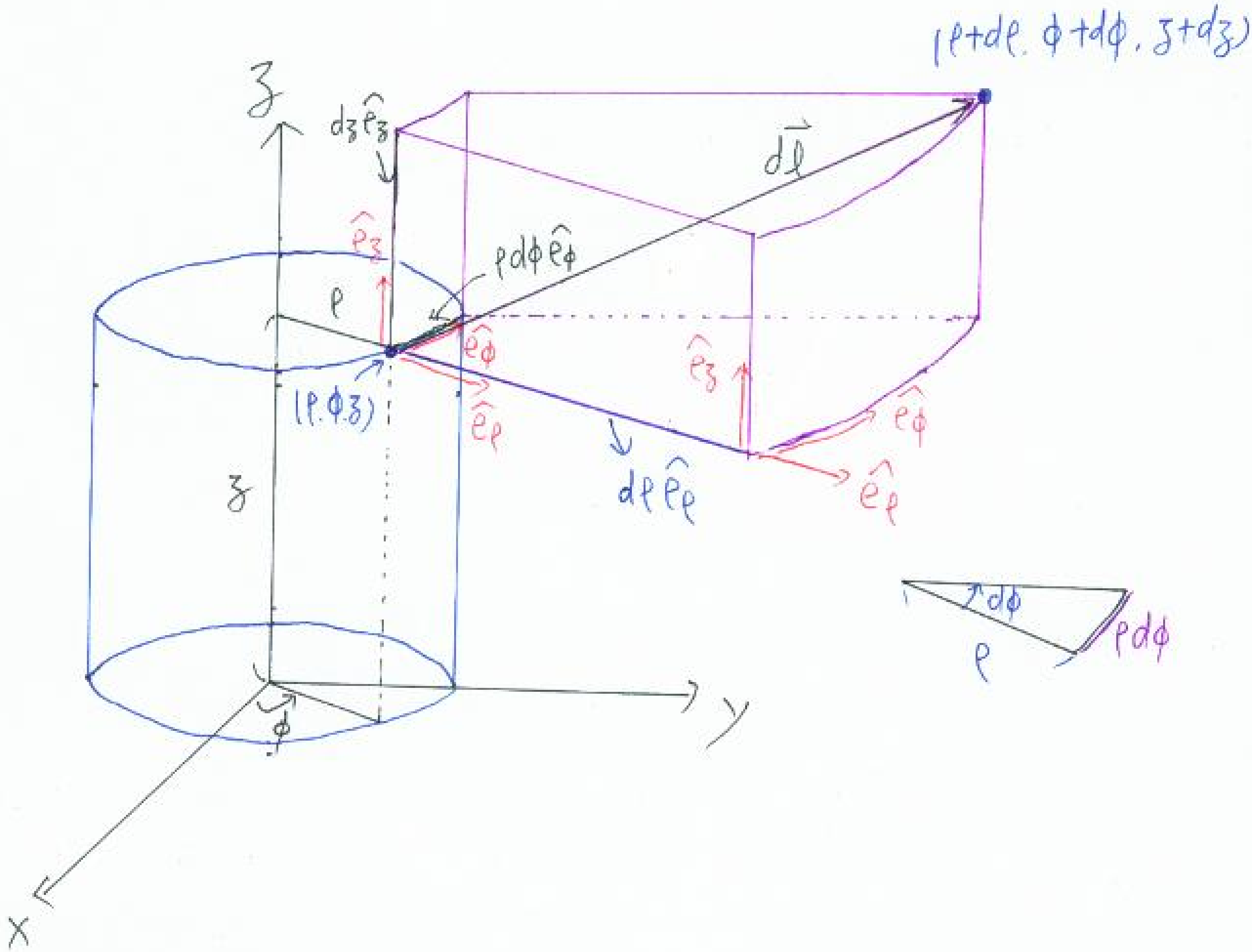
§ 直角座標



座標	x	y	z
基底	$\hat{e}_x$	$\hat{e}_y$	$\hat{e}_z$
尺度因子	1	1	1

$$d\vec{l} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$$

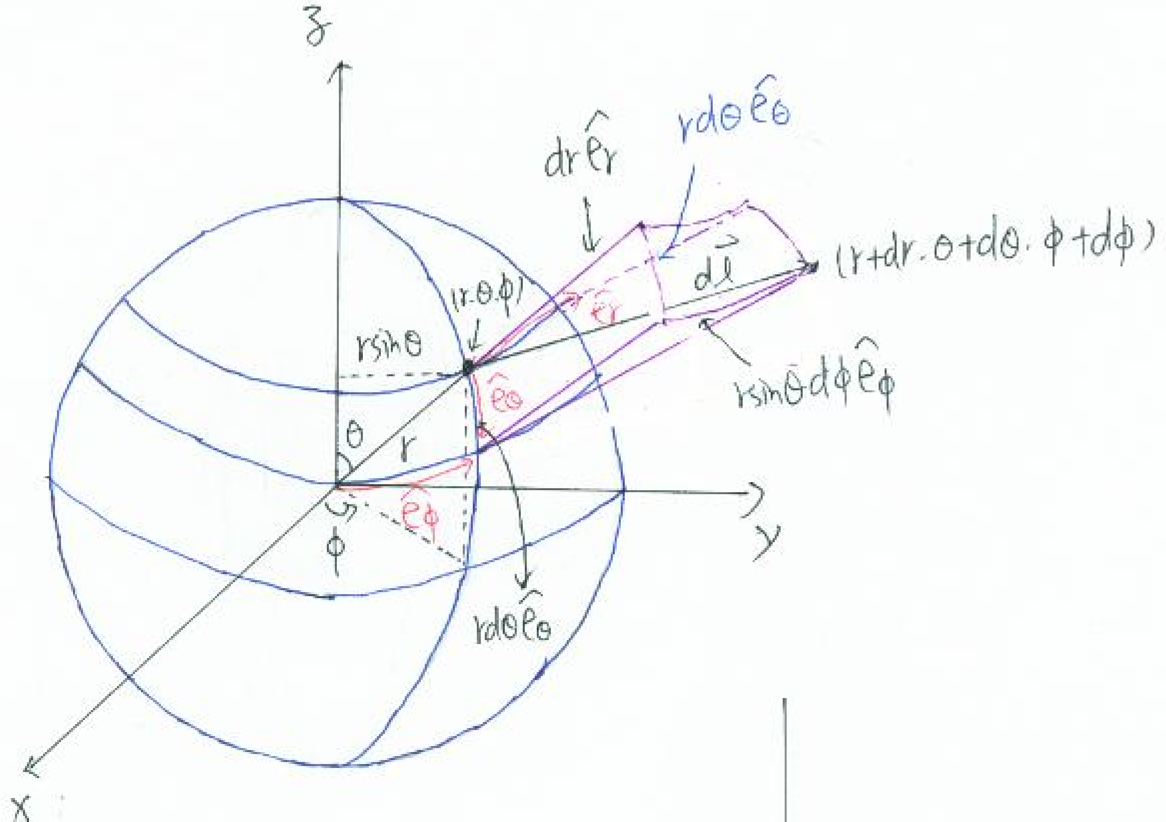
§柱座標

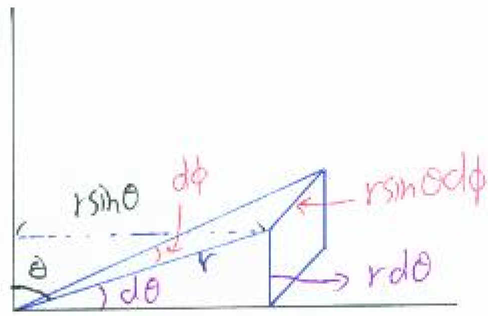


$$d\vec{l} = d\rho\,\hat{e}_\rho + \rho\, d\phi\,\hat{e}_\phi + d z\,\hat{e}_z$$

座標	ρ	ϕ	z
基底	$\hat{e}_\rho$	$\hat{e}_\phi$	$\hat{e}_z$
尺度因子	1	ρ	1

§球座標





$$d\vec{l} = dr\hat{e}_r + r d\theta \hat{e}_\theta + r\sin\theta d\phi \hat{e}_\phi$$

座標	r	θ	ϕ
基底	$\hat{e}_r$	$\hat{e}_\theta$	$\hat{e}_\phi$
尺度因子	1	r	$r\sin\theta$

§ 梯度

曲線式：$\vec{\nabla}\phi = \hat{e}_1 \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}$

基本式：$(\vec{\nabla}\phi)\cdot d\vec{l} = d\phi$

推導：

$$\begin{aligned}
d\phi &= \phi(q_1+dq_1, q_2+dq_2, q_3+dq_3) - \phi(q_1, q_2, q_3) \\
&= \frac{1}{dq_1}[\phi(q_1+dq_1, q_2+dq_2, q_3+dq_3) - \phi(q_1, q_2+dq_2, q_3+dq_3)]\,dq_1 \\
&\quad + \frac{1}{dq_2}[\phi(q_1, q_2+dq_2, q_3+dq_3) - \phi(q_1, q_2, q_3+dq_3)]\,dq_2 \\
&\quad + \frac{1}{dq_3}[\phi(q_1, q_2, q_3+dq_3) - \phi(q_1, q_2, q_3)]\,dq_3 \\
&= \frac{\partial\phi}{\partial q_1}dq_1 + \frac{\partial\phi}{\partial q_2}dq_2 + \frac{\partial\phi}{\partial q_3}dq_3 \quad \text{—— ①}
\end{aligned}$$

$$\begin{aligned}
(\vec{\nabla}\phi)\cdot d\vec{l} &= [\hat{e}_1(\vec{\nabla}\phi)_1 + \hat{e}_2(\vec{\nabla}\phi)_2 + \hat{e}_3(\vec{\nabla}\phi)_3]\cdot[\hat{e}_1 h_1 dq_1 + \hat{e}_2 h_2 dq_2 + \hat{e}_3 h_3 dq_3] \\
&= (\vec{\nabla}\phi)_1 h_1 dq_1 + (\vec{\nabla}\phi)_2 h_2 dq_2 + (\vec{\nabla}\phi)_3 h_3 dq_2 \quad \text{—— ②}
\end{aligned}$$

$\because$ ① = ②

$\therefore (\vec{\nabla}\phi)_1 + (\vec{\nabla}\phi) + (\vec{\nabla}\phi)_3 = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}$

$\Rightarrow \vec{\nabla}\phi = \hat{e}_1 \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}$

直角座標：

x	y	z
$\hat{e}_x$	$\hat{e}_y$	$\hat{e}_z$
1	1	1

$\Rightarrow \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$

球座標：

怕跟純量ϕ符號一樣，故改成φ

r	θ	φ
$\hat{e}_r$	$\hat{e}_\theta$	$\hat{e}_\varphi$
1	r	$r\sin\theta$

$\Rightarrow \vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$

柱座標：

ρ	φ	z
$\hat{e}_\rho$	$\hat{e}_\varphi$	$\hat{e}_z$
1	ρ	1

$\Rightarrow \vec{\nabla}\phi = \frac{\partial\phi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi + \frac{\partial\phi}{\partial z}\hat{e}_z$

§散度

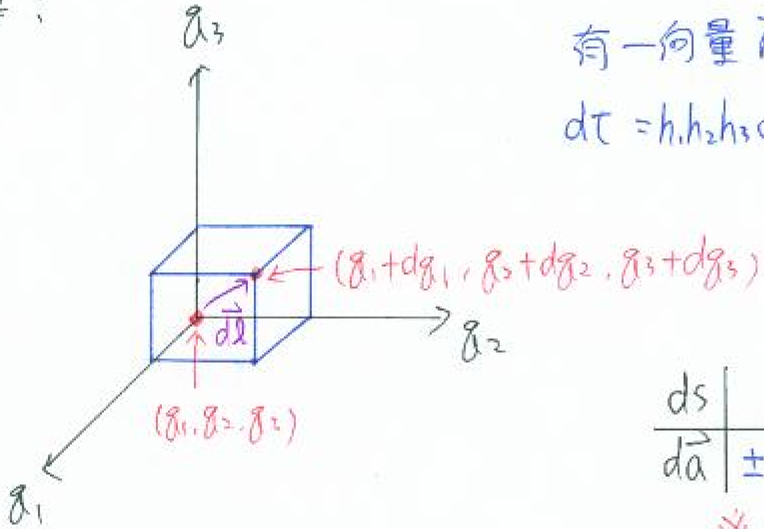
曲線式：$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_1h_3A_2) + \frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$

基本式：$\vec{\nabla}\cdot\vec{A} = \frac{1}{d\tau}\oint_{ds}\vec{A}\cdot d\vec{a}$

$\int_V(\vec{\nabla}\cdot\vec{v})d\tau = \oint_S \vec{v}\cdot d\vec{a}$ 代表單一封閉面

$\vec{\nabla}\cdot\vec{v}\,d\tau = \oint_{ds}\vec{A}\cdot d\vec{a}$ 代表由多個面組成一封閉面

推導：



有一向量 $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$d\tau = h_1h_2h_3\,dq_1\,dq_2\,dq_3$

ds	上下	左右	前後
$d\vec{a}$	$\pm h_1h_2\,dq_1\,dq_2\,\hat{e}_3$	$\mp h_1h_3\,dq_1\,dq_3\,\hat{e}_2$	$\pm\hat{e}_1 h_2h_3\,dq_2\,dq_3$

※共六個面

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{d\tau}\oint_{ds}\vec{A}\cdot d\vec{a}$$

$$= \frac{1}{h_1h_2h_3\,dq_1\,dq_2\,dq_3}\Big\{\frac{1}{dq_3}\left[(h_1h_2A_3)_{q_3+dq_3} - (h_1h_2A_3)_{q_3}\right]dq_1\,dq_2\,dq_3$$

$$+ \frac{1}{dq_2}\left[-(h_1h_3A_2)_{q_2} + (h_1h_3A_2)_{q_2+dq_2}\right]dq_1\,dq_3\,dq_2$$

$$+ \frac{1}{dq_1}\left[(h_2h_3A_1)_{q_1+dq_1} - (h_2h_3A_1)_{q_1}\right]dq_2\,dq_3\,dq_1\Big\}$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_1h_3A_2) + \frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$$

直角座標：$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

球座標：$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial\phi}(rA_\phi)\right]$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(A_\phi)$$

柱座標：$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(\rho A_z)\right]$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$$

§ 純量場的 Laplacian

曲線式：$\vec{\nabla}^2\phi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right.$

基本式：$\vec{\nabla}^2\phi = \vec{\nabla}\cdot(\vec{\nabla}\phi)$

直：

x	y	z
$\hat{e}_x$	$\hat{e}_y$	$\hat{e}_z$
1	1	1

$$\Rightarrow \vec{\nabla}^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

柱：

ρ	φ	z
$\hat{e}_\rho$	$\hat{e}_\varphi$	$\hat{e}_z$
1	ρ	1

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\varphi}\frac{\partial\phi}{\partial\varphi} + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

球

r	θ	ϕ
$\hat{e}_r$	$\hat{e}_\theta$	$\hat{e}_\phi$
1	r	$r\sin\theta$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\phi}{\partial\phi}\right.$$
$$= \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial\phi}{\partial\varphi}\right)$$

§旋度

曲線式：$\vec{\nabla}\times\vec{A}=\dfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial g_1} & \frac{\partial}{\partial g_2} & \frac{\partial}{\partial g_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

$$=\frac{\hat{e}_1}{h_2h_3}\left\{\frac{\partial}{\partial g_2}(h_3A_3)-\frac{\partial}{\partial g_3}(h_2A_2)\right\}+\frac{\hat{e}_2}{h_1h_3}\left\{\frac{\partial}{\partial g_3}(h_1A_1)-\frac{\partial}{\partial g_1}(h_3A_3)\right\}$$
$$+\frac{\hat{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial g_1}(h_2A_2)-\frac{\partial}{\partial g_2}(h_2A_1)\right\}$$

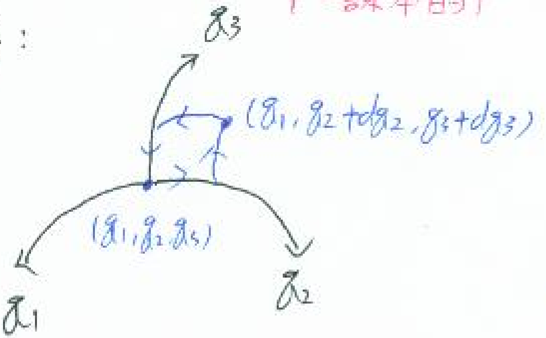
基本式：$\vec{\nabla}\times\vec{A}=\dfrac{1}{da}\oint_{d\Gamma}\vec{A}\cdot d\vec{l}$

Γ=課本的P

$\int\vec{\nabla}\times\vec{A}\cdot d\vec{a}=\oint_{\Gamma}\vec{A}\cdot d\vec{l}$ 代表單一封閉迴路

$\vec{\nabla}\times\vec{A}\cdot d\vec{a}=\oint_{d\Gamma}\vec{A}\cdot d\vec{l}$ 代表由多條線組成的封閉迴路

推導：



$d\Gamma$	上下	左右
$d\vec{l}$	$\mp\hat{e}_2h_2dg_2$	$\mp\hat{e}_3h_3dg_3$

$\vec{A}=A_1\hat{e}_1+A_2\hat{e}_2+A_3\hat{e}_3$

$d\vec{a}=\hat{e}_1h_2h_3dg_2dg_3$

有3個方向的面$(\hat{e}_1,\hat{e}_2,\hat{e}_3)$，選$\hat{e}_1$來推導

$$(\vec{\nabla}\times\vec{A})_1\cdot\hat{e}_1h_2h_3dg_2dg_3=\frac{1}{dg_3}\left[-(h_2A_2)_{g_3+dg_3}+(h_2A_2)_{g_3}\right]dg_2dg_3$$
$$+\frac{1}{dg_2}\left[-(h_3A_3)_{g_2}+(h_3A_3)_{g_2+dg_2}\right]dg_3dg_2$$
$$=-\frac{\partial}{\partial g_3}(h_2A_2)dg_2dg_3+\frac{\partial}{\partial g_2}(h_3A_3)dg_3dg_2$$

$\Rightarrow$ $(\vec{\nabla}\times\vec{A})_1=\dfrac{\hat{e}_1}{h_2h_3}\left\{\dfrac{\partial}{\partial g_2}(h_3A_3)-\dfrac{\partial}{\partial g_3}(h_2A_2)\right\}$

同理：

$$(\vec{\nabla}\times\vec{A})_2=\frac{\hat{e}_2}{h_1h_3}\left\{\frac{\partial}{\partial g_3}(h_1A_1)-\frac{\partial}{\partial g_1}(h_3A_3)\right\}$$
$$(\vec{\nabla}\times\vec{A})_3=\frac{\hat{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial g_1}(h_2A_2)-\frac{\partial}{\partial g_2}(h_1A_1)\right\}$$

故：$\vec{\nabla}\times\vec{A}=(\vec{\nabla}\times\vec{A})_1+(\vec{\nabla}\times\vec{A})_2+(\vec{\nabla}\times\vec{A})_3$

$$=\frac{\hat{e}_1}{h_2h_3}\left\{\frac{\partial}{\partial g_2}(h_3A_3)-\frac{\partial}{\partial g_3}(h_2A_2)\right\}+\frac{\hat{e}_2}{h_1h_3}\left\{\frac{\partial}{\partial g_3}(h_1A_1)-\frac{\partial}{\partial g_1}(h_3A_3)\right\}$$
$$+\frac{\hat{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial g_1}(h_2A_2)-\frac{\partial}{\partial g_2}(h_1A_2)\right\}$$
$$=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial g_1} & \frac{\partial}{\partial g_2} & \frac{\partial}{\partial g_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

再視座標將尺度因子代入即可