$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial z} = \varepsilon_0 \left(E_r \cos\theta - E_\theta \sin\theta\right)_{r=r'} = -\frac{qd}{4\pi r^3}$$

The total charge Q at $r=r'$, $z=-d/2$

$$Q = \int \sigma \cdot 2\pi r\,dr = -q$$

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§ 3-3 Method of Variable Separation (分離變數法)

In rectangular coordinates:

The Laplace's equation can be expressed as

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V = 0$$

If the solution as the produce of three equations.

$$V(x,y,z) = X(x) \cdot Y(y) \cdot Z(z)$$

then $\nabla^2 V = \nabla^2 (X(x) \cdot Y(y) \cdot Z(z))$

$$\nabla^2 XYZ = \frac{\partial^2}{\partial x^2} XYZ + \frac{\partial^2}{\partial y^2} XYZ + \frac{\partial^2}{\partial z^2} XYZ$$
$$= YZ \frac{\partial^2}{\partial x^2} X + XZ \frac{\partial^2}{\partial y^2} Y + XY \frac{\partial^2}{\partial z^2} Z$$

同除以 XYZ

$$\frac{1}{XYZ} \nabla^2 XYZ = \frac{1}{X} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y + \frac{1}{Z} \frac{\partial^2}{\partial z^2} Z = 0$$

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Then we can change the above function as

$$\boxed{\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = \frac{-1}{Z}\frac{\partial^2}{\partial z^2}Z}$$

to be valid for arbitary values of independent coordinates.

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -C_3 \quad \text{if } \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z = C_3$$

Similarly

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y - C_3 = C_1$$

then

$$\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = \boxed{-C_3 - C_1}$$

The forms of solution of Laplaces

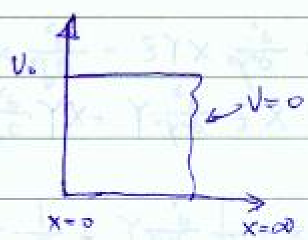
(1) $\frac{\partial^2}{\partial x^2}X_{(x)} = C_1 X_{(x)}$

A. $X_{(x)} = A\cdot e^{-kx}$ or $A\cdot e^{kx}$

B. $X_{(x)} = A\sin kx$ or $A\cos kx$

C. $X_{(x)} = A\cdot\sinh x_{(x)}$ or $A\cosh h_{(x)}$

(2) Considering of Boundary conditions.



$A\cdot e^{kx} + Be^{-kx}$:

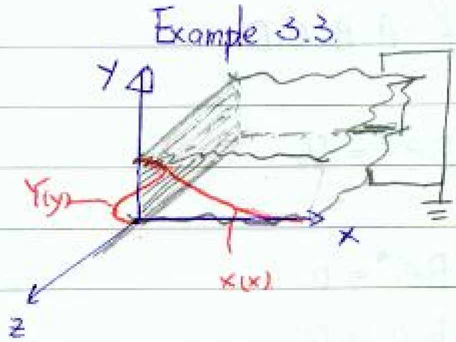
$\Rightarrow$ if $x=\infty$, $V=0$

$X_{(\infty)} = A\cdot\infty + B\cdot 0 = 0$

$\Rightarrow A=0,\ B\neq 0$

$X_{(x)} = B\cdot e^{-kx}$

Example 3.3.



Two infinite grounded metal plates lie parallel to the XZ plane. one at Y=0, the other at y=a. The left end at x=0, maintained as a potential $V_0(y)$

1. BC condition

$\pm z \to \infty$

2D Laplace's eq.

$V(x,y) = X(x) \cdot Y(y)$

a. V=0, when y=0
b. V=0, " y=a
c. $V=V_0$, " x=0
d. V=0, " $x=\infty$

⇒ Find the forms of solution

$$V(x,y) = X(x) \cdot Y(y) \Rightarrow \nabla^2 V = 0$$

$$\frac{1}{X}\frac{d^2}{dx^2}X = C_1, \quad \frac{1}{Y}\frac{d^2}{dy^2}Y = C_2, \quad C_1 + C_2 = 0$$

if y=0, y=a, V=0.
So the solution for Y(y) can be expressed as

$$Y(y) = C\sin ky + D\cos ky$$

if x=0. $V(0) = V_0$.
$x=\infty$, $V(\infty) = 0$

the solution is $X(x) = A \cdot e^{kx} + Be^{-kx}$

Find the coefficients of A.B.C.D

1. If $x=0$

$$X(0) = A\cdot e^{0} + Be^{-0} = V_0$$

If $x=\infty$

$$X(\infty) = A\cdot e^{\infty} + Be^{-\infty} = 0$$

So that $A\cdot\infty + B\cdot 0 = 0$

So the $A=0$, then $B\neq 0$

the solution of $X(x)$. $\boxed{X(x) = Be^{-kx}}$

If $y=0$

$$Y(0) = C\cdot\sin(k\cdot 0) + D\cdot\cos(k\cdot 0) = 0$$

$$\Rightarrow \quad C\cdot 0 + D\cdot 1 = 0$$

then $C\neq 0$, $D=0$

$$Y(y) = C\cdot\sin ky$$

If $y=a$

$Y(a) = C\sin ka = 0$, so the $k=\frac{n\pi}{a}$

The final form $\boxed{Y(y) = C\sin\frac{n\pi}{a}y}$, $n=1,2,3\ldots$

So the multi-solution is

$$\begin{bmatrix} V_1(x,y) = C_1 e^{\frac{\pi}{a}x}\cdot\sin\frac{\pi}{a}y \\ V_2(x,y) = C_2 e^{\frac{2\pi}{a}x}\cdot\sin\frac{2\pi}{a}y \\ \vdots \qquad\qquad \vdots \\ V_n(x,y) = C_n\cdot e^{\frac{n\pi}{a}x}\cdot\sin\frac{n\pi}{a}y \end{bmatrix} \Rightarrow \boxed{V(x,y) = \sum_{n=1}^{\infty} C_n\cdot e^{-\frac{n\pi}{a}x}\cdot\sin\frac{n\pi}{a}y}$$

C_n = coefficient

* If $x=0$

$$V(0,y) = \sum C_n\cdot e^{-0}\cdot\sin\frac{n\pi}{a}y = V_0(y)$$

Special skill : Integral of Both sides.

$$\sum_{n=1}^{\infty} C_n \cdot \int_0^a \sin\frac{n\pi}{a}y \cdot \sin\frac{n'\pi}{a}y\, dy = \int V_0(y) \cdot \sin\frac{n'\pi}{a}y\, dy$$

The limits of y are 0 to a

$$\int_0^a \sin\frac{n\pi}{a}y \cdot \sin\frac{n'\pi}{a}y\, dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases}$$

$$\Rightarrow C_n \cdot \frac{a}{2} = \int V_0(y) \cdot \sin\frac{n\pi}{a}y\, dy$$

So the $C_n = \frac{2}{a} V_0 \int_0^a \sin\frac{n\pi}{a}y\, dy$

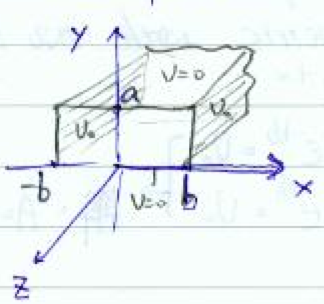
$$\Rightarrow C_n = \frac{2}{a} V_0 \left(\frac{a}{n\pi}\right) \cdot (1 - \cos n\pi) \quad \begin{cases} n = \text{even} & \cos n\pi = 1 & C_n = 0 \\ n = \text{odd} & \cos n\pi = -1 & C_n = \frac{4V_0}{n\pi} \end{cases}$$

$$\Rightarrow V(x,y) = \sum_n \frac{4V_0}{n\pi} \cdot \sin\frac{n\pi}{a}y \cdot e^{-\frac{n\pi}{a}x} \qquad n = 2.4.6 \cdots$$

if $x = 0 \rightarrow V = V_0 \Rightarrow X = A \cdot e^{-n\pi/a \cdot x}$

$x = \infty \rightarrow V = 0$

Example 3.4



$x = b, \; V = V_0 \longrightarrow e^{-kx}$
$x = -b, \; V = V_0 \longrightarrow e^{kx}$
$\Big|$ $\cosh kx$, $\sinh kx$

So the solution is a form of

$$X = A \cosh kx + B \sinh kx$$

Example: if the potential is displayed below.

Example: 3.4.

B.C. ① $x=b \quad V=V_0$

$x=-b \quad V=V_0$

② $y=0 \quad V=0$

$y=a \quad V=0$

Let $V(x,y) = X(x)\cdot Y(y)$ 代入 Laplace's equation.

$$\frac{1}{X}\cdot\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\cdot\frac{\partial^2}{\partial y^2}Y = 0$$

if $\frac{1}{X}\cdot\frac{\partial^2}{\partial x^2}X = C_1$. then $\frac{1}{Y}\cdot\frac{\partial^2}{\partial y^2}Y = -C_1$

if $\frac{\partial^2}{\partial x^2}X = C_1\cdot X$

$$\begin{cases} e^{kx} & 微分兩次 & \pm k^2 e^{kx} \\ \sin kx & = & \pm k^2 \sin kx \\ \cos kx & = & \pm k^2 \cos kx \\ e^{-kx} & = & \pm k^2 e^{-kx} \end{cases}$$

The forms of solution of $X(x)$. Because the x-axis is symmetric with two B.C.

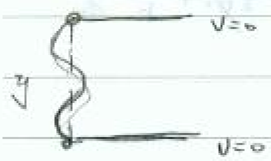
$$X(x) = A e^{kx} + B e^{-kx}$$

$$\left.\begin{cases} x=b & A e^{kb} + B e^{-kb} = V_0 \\ x=-b & A e^{-kb} + B e^{kb} = V_0 \end{cases}\right\} \rightarrow 解: A=B$$

So the solution $X(x)$

$$X(x) = A[e^{kx}+e^{-kx}] = \frac{A'}{2}[e^{kx}+e^{-kx}] = \boxed{A' \cosh(kx)}$$

Let the B.C. of $Y(y)$



$$Y(y) = C \sin ky + D \cos ky$$

$$y=0. \quad Y(0) = C \cdot \sin 0 + D \cdot 1 = 0$$

$$D = 0. \quad C \neq 0$$

$$Y(y) = C \sin ky. \quad k = \frac{n\pi}{a}$$

The total solution.

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cdot \cosh kx \cdot \sin ky = \sum_{n=1}^{\infty} C_n \cdot \cosh \frac{n\pi}{a}x \cdot \sin \frac{n\pi}{a}y$$

* Find the coefficient

if $x=b$ A' λ.

$$V_0 = \sum_n C_n \cdot \cosh \frac{n\pi b}{a} \cdot \sin \frac{n\pi}{a}y$$

Special skill

Integral Both sides

$$\int V_0 \cdot \sin \frac{n\pi}{a}y \, dy = \int C_n \cosh \frac{n\pi}{a}b \cdot \sin \frac{n\pi}{a}y \cdot \sin \frac{n'\pi}{a}y \, dy$$

This solution is $C_n \cdot \cosh(\frac{n\pi}{a}b) = \begin{cases} 0 & n \text{ is even} \\ \frac{4V_0}{n\pi} & n \text{ is odd.} \end{cases}$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{odd} \frac{1}{n} \cdot \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \cdot \sin \frac{n\pi}{a}y$$

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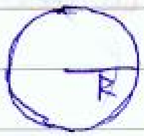
* Potential of a uniform sphere of charge.

Poisson's equation.



$Q = \frac{4}{3}\pi R^3 \cdot \rho$

In spherical polar coordinates

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\varphi^2} + \frac{2}{r}\frac{d}{dr} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)V = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = -\frac{\rho}{\varepsilon_0}$$

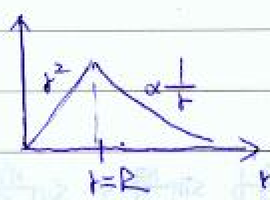
if $\rho = 0$, the solution of : $\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = 0$

$$V = \frac{a}{r} + b$$

$$\Rightarrow \nabla_r^2 V = 0$$

① Since the zero potential is arbitrary
This gives the values $b = 0$. $[V(r) = \frac{a}{r} + b]$

② Since the sphere of charge will look like a point charge at large distance.



$a = \frac{Q}{4\pi\varepsilon_0}$ $\quad E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}$

$V = \frac{Q}{4\pi\varepsilon_0 r}$

$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r}$ outside the sphere