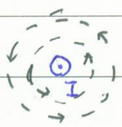
No.
Date. Mar. 18

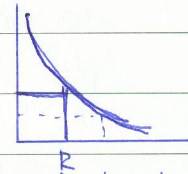
## § 5.3 The divergence of $\vec{B}$

### § 5.3.1 straight-line currents (Ampere's world of view)

↳ circuit law

$$B = \frac{\mu_0 I}{2\pi R}$$





It's clear that the fields has a non-zero curl for closed path (without work done)

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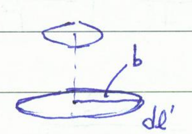
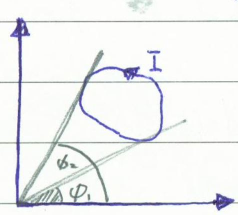
According to 5.36 the integral of $\vec{B}$ around a circuit path of radius b.

$\vec{B} \equiv \frac{\mu_0 I}{2\pi s}$ +

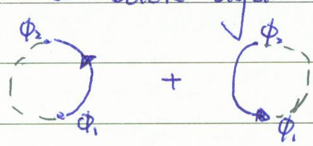


$\Rightarrow \boxed{\oint \frac{\mu_0 I}{2\pi s}\, dl} \overset{?}{\Rightarrow} \mu_0 I$



if a current around a closed path of $\Gamma$

if the $\phi$ would go from $\phi_1$ to $\phi_2$
& back again $\phi_1 \to \phi_2$



= closed path.

Then the line integral will be

$$\oint \vec{B}\cdot d\vec{l} = \int_{\phi_1}^{\phi_2} \vec{B}\cdot d\vec{l} + \int_{\phi_2}^{\phi_1} \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

by applying.

Applying to Stoke's theorem to

curl $\vec{B}\cdot d\vec{l}$

$d\vec{a}$

$\frac{\vec{B}\cdot d\vec{l}}{d\vec{a}}$

$$\int \vec{B}\cdot d\vec{l} = \mu_0 I = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$\Downarrow$$

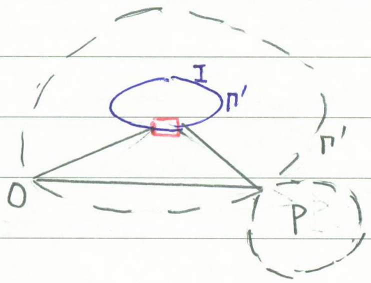
$$\int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$\Downarrow$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Note Junction to 5.3 of curl of magnetic field. We need to prove the theorem of Ampere.

According to Biot-Savarc Law in a closed path of current I, in point P. We can get expression as




at point P's magnetic field

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{\Gamma} \frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

At point P around a closed path of Γ

$$\oint \vec{B}_P\cdot d\vec{l}_P = \begin{cases} 0 & ,\ \Gamma \text{ didn't cover } \Gamma' \\ \mu_0 I & ,\ \Gamma \text{ cover } \Gamma' \end{cases}$$

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The integral function of $\Gamma$ can be obtained

$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{4\pi}\oint_{\Gamma}\left[\oint_{\Gamma'}\frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]\cdot d\vec{l}$$

Prove the results of $\int_{\Gamma}\int_{\Gamma'}\frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot dl' = \begin{cases} 0 & \text{NO!} \\ 4\pi & \text{YES!}\end{cases}$

Note : In electric field if we set Gradient of $\frac{1}{|\vec{r}-\vec{r}'|}$, we can write eq.

$$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\nabla_r\frac{1}{|\vec{r}-\vec{r}'|} = \nabla_{r'}\frac{1}{|\vec{r}-\vec{r}'|}$$

Let the integral function can be represent as

$$\oint_{\Gamma}\oint_{\Gamma'} d\vec{l}'\times\nabla_{r'}\left(\frac{1}{\vec{r}-\vec{r}'}\right)\cdot dl'$$

$\Downarrow$

(vector x vector) dot a vector = scale

where $\oint(d\vec{l}\times\vec{f})$

$$\oint_{\Gamma} d\vec{l}\times\vec{f} = \oint_S \nabla(\vec{f}\cdot d\vec{s}) - \int d\vec{s}\,(\vec{\nabla}\cdot\vec{f})$$

$$\nabla(\vec{f}\cdot d\vec{s}) = (d\vec{s}\cdot\nabla)\vec{f} + d\vec{s}\times(\vec{\nabla}\times\vec{f})$$

$$\oint d\vec{l} \times \vec{f} \cdot d\vec{l} = \oint (d\vec{s} \cdot \nabla) \vec{f} \cdot d\vec{l} + \int d\vec{s} \times (\nabla \times \vec{f}) \cdot d\vec{l} - \int d\vec{s}\, (\nabla \cdot \vec{f}) \cdot d\vec{l}$$

$$= \oint_{\Gamma'} \oint_{s} d\vec{s}' \cdot \nabla' \left[ \nabla' \frac{1}{|\vec{r}-\vec{r}'|} \right] \cdot d\vec{l}$$

$$+ \oint_{\Gamma} \oint_{s'} d\vec{s}' \times \left( \nabla' \times \left( \nabla' \frac{1}{|\vec{r}-\vec{r}'|} \right) \right) \cdot d\vec{l}$$

$$- \oint_{\Gamma} \oint_{s'} \nabla' \cdot \left( \nabla' \frac{1}{|\vec{r}-\vec{r}'|} \right) d\vec{s} \cdot d\vec{l}$$

The 1-st term

$$d\vec{s}' \cdot \nabla' \left( \nabla' \frac{1}{|\vec{r}-\vec{r}'|} \right) \cdot d\vec{l} = - d\vec{s} \cdot \nabla \left( \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right) \cdot d\vec{l}$$

$$= - d\vec{s} \cdot \frac{d\left( \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right)}{dl} \cdot d\vec{l}$$

$$\Rightarrow \oint (-d\vec{s}') \cdot d\left( \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right) = - d\vec{s}' \cdot \oint_{\Gamma} d\left( \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right) = 0$$

The 2nd term

$$\nabla' \times \nabla' \left( \frac{1}{|\vec{r}-\vec{r}'|} \right) = 0 \qquad \nabla \times \vec{J} = 0$$

$$\oint_{\Gamma} \left( \nabla' \times \nabla' \frac{1}{|\vec{r}-\vec{r}'|} \right) dl = 0$$

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The Last term

$$\oint_{\Gamma}\oint_{\Gamma'} d\vec{l}' \times \nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) d\vec{s}' \cdot d\vec{l}$$

$$= - \oint_{\Gamma}\oint_{s'} \nabla' \cdot \left(\nabla' \frac{1}{|\vec{r}-\vec{r}'|}\right) d\vec{s}' \cdot d\vec{l}$$

If $\nabla' \cdot \nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \nabla'^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}')$ Dirac function

Let be

$$= - \oint_{\Gamma}\oint_{s'} (-4\pi\delta(r-r')) d\vec{s}' \cdot d\vec{l}$$

$$= 4\pi \oint_{\Gamma}\oint_{s'} \delta(\vec{r}-\vec{r}') d\vec{s} \cdot d\vec{l} =$$

$$= \left[ \begin{array}{ll} \text{If } r' = r & \text{then } \oint_{\Gamma}\oint_{s} \delta(r-r') ds' dl' = 4\pi \\ \text{If } r' \neq r & 0 \end{array} \right.$$

$$\int \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{4\pi} \cdot 4\pi = \mu_0 I$$

Review of Biot-Savart Law to the curl of B-field

1. $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$
2. $I = \int \vec{J}\cdot da$
3. $\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot da$
4. Stoke's theorem

$$\oint \nabla\times\vec{B}\cdot da = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$\Rightarrow \nabla\times\vec{B} = \mu_0\vec{J}$$

§ 5.3.2

Biot-Savart Law to the divergency of B-field

From the point of view: or Reverse Divergency

(1) Line current: $\vec{B}(r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}$

(2) surface current: $\vec{B}(r) = \frac{\mu}{4\pi}\int \frac{\vec{K}(r')\times\hat{r}}{r^2}da'$

(3) volume current: $\vec{B}(r) = \frac{\mu}{4\pi}\int \frac{\vec{J}(r')\times\hat{r}}{r^2}dv'$

$$\text{Vector} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\hat{r}}{r^2} = -\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

向量 = 純量 函數 的梯度

$$B(r) = \frac{\mu_0}{4\pi}\int \vec{J}(r')\times\frac{\hat{r}}{r^2}\,dr'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}(r')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d^3r'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}(r')\times\left[-\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

$$\vec{A}'\times(-\nabla f)$$

$$= \frac{\mu_0}{4\pi}\int \vec{\nabla}_r\times\left[\frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|}\right]d^3r'$$

Using product rule:

(1) $\nabla\times(f\vec{A}') = f(\nabla\times\vec{A}') - \vec{A}'\times(\nabla f)$

(2) $\nabla\times\vec{A}' = \nabla\times\vec{J} = 0$

(3) $\nabla\times(f\vec{A}') = -\vec{A}'\times(\nabla f)$

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So the Biot-Savart Law of volume current can be reduced as:

$$B(\vec{r}) = \frac{\mu_0}{4\pi}\int \nabla_r \times \left[\frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] d^3r'$$

$$= \frac{\mu_0}{4\pi}\int \nabla_r \times \vec{A}\, d^3r'$$

Vector identify: $\nabla_r \cdot (\nabla_r \times \vec{A}(\vec{r},\vec{r}')) = 0$

$$\nabla_r \cdot \vec{B} = \nabla_r \cdot (\nabla_r \times \vec{A}) = 0$$

Applying to curl of $\vec{B}$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0 \vec{J}$$

§ 5.3.3. Applications of Ampere's Laws

(1) Enclosed current is the key point uniform current.

a. If point p is $r > R$

$\Rightarrow$ enclosed current $I_{enc}$ i.e. $I_0$ $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_0$

b. If point p inside $r < R \Rightarrow$ enclosed current

$\left.\begin{array}{l}\text{Total volume is } \pi R^2 L \\ \text{enclosed volume is } \pi r^2 L\end{array}\right]$ then $I_{enc} = \frac{r^2}{R^2} I_0$

$\oint \vec{B}\cdot d\vec{l} = \mu_0 \frac{r^2}{R^2} I_0$

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The line integral of $\vec{B}$ will be $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$I_{enc}$ = total current enclosed by the loop is represent by a volume current density $\vec{J}$.

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

with the integral take over the surface bounded by the loop of Applying Stoke's theorem

$$\int (\nabla \times \vec{B}) \cdot da = \mu_0 \int \vec{J} \cdot d\vec{a}$$