Econometric Analysis
Midterm

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There are 110% points with extra 10% as a bonus. Yet, the highest grade is still 100%.

1 Basic concepts (10%)

Define and explain the following terms.

1. skewness of a random variable
2. Conditional variance
3. law of iterated expectation
4. converge in probability
5. p-value

2 Matrix (10%)

Let

$$A = \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Determine

1. A - 2D
2. rank of AD
3. $|D|$
4. A^{-1}
5. eigen values of D

3 Evaluating probabilities (10%)

1. Let (X, Y) be standard bivariate normal random variables. What is $P(X = 0)$?
2. Z be normal with mean 1 and variance 4. Evaluate $P(-4.16 < Z < 4.92)$. (Note that $4.16 = 2.58 * 2 - 1, 4.92 = 1.96 * 2 + 1$).
3. Let the accidents Y per week occurring at the intersection follows a Poisson distribution with mean 5. Find the probability that there are less than 5 incidents occurring next month. Writing down the formula is good enough and numerical calculation is not required.
4. For the Taiwan lottery, one selects six regular and one special numbers (without replacement) from numbers 1 to 49. Matching all six regular numbers wins the first prize while matching any 5 of 6 regular numbers and the special number wins the second prize. Assume that the winning numbers are randomly drawn. Compute the probability of any lottery ticket to win the first and second prize respectively.
5. For the World Series, two teams, say Yankee and Red Sox, play a series of games until one wins four games. Assume that the probability that Yankee wins a game is .55. Assume that all games are independently played. Compute the probability that the series ends at 6 games (4 wins and 2 losses).

4 Unconditional and Conditional (10%)

Let $Z, \epsilon_1, \epsilon_2$ be standard normal independent of each other. Let

$$\begin{aligned} X &= 2Z + \epsilon_1 \\ Y &= -2Z + \epsilon_2 \end{aligned}$$

Answer the following questions:

1. Find $Cov(X, Y)$. Are X, Y correlated or uncorrelated?
2. Find $Cov(X, Y|Z)$. Conditional upon Z, are X, Y correlated or uncorrelated? Even if you can not prove your claim, make an educated guess and explain your answer as clearly as possible.

5 Estimation (10%)

Let $Y_1, \ldots, Y_n$ denote a random sample from a population with mean μ and variance σ^2. Consider the following these estimators for μ:

$$\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2), \quad \hat{\mu}_2 = \frac{1}{4}Y_1 + \frac{Y_2 + \ldots + Y_{n-1}}{2(n-2)} + \frac{1}{4}Y_n, \quad \hat{\mu}_3 = \bar{Y}$$

a. Show that each of three estimators is unbiased.

b. Find the efficiency of $\hat{\mu_3}$ relative to $\hat{\mu_2}$ and $\hat{\mu_1}$, respectively.

6 Least squares regression (20%)

Let the regression model be $X_i = \alpha_0 + \alpha_1 Y_i + v_i$, where v_i are iid's with mean 0 and variance σ^2.

1. Derive the least square estimates of $\alpha_0, \alpha_1, \sigma^2$, denoted as $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\sigma}^2$.
2. Assume that Y is fixed regressor. Show that $\hat{\alpha}_1$ can be expressed as a weighted averages of X_i and find the weights.
3. Find $E(\hat{\alpha}_1)$ and $var(\hat{\alpha}_1)$.

7 Estimating the slope (15%)

Let y_t be generated by:

$$y_t = \alpha + \beta x_t + \epsilon_t,\ \ t = 1, \cdots, T \tag{1}$$

where ϵ_t is *iid* with mean 0 and variance σ^2, $\alpha = -1, \beta = 2$, and x_t is fixed regressor with values given in each case below.

- Case 1:
  $x_t = (-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5)', \sigma^2 = 1, T = 11$
- Case 2:
  $x_t = (-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5)', \sigma^2 = 2, T = 11$
- Case 3:
  $x_t = (-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10)', \sigma^2 = 1, T = 11$
- Case 4:
  $x_t = (-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10)', \sigma^2 = 2, T = 11$
- Case 5:
  $x_t = (-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0)', \sigma^2 = 1, T = 11$
- Case 6:
  $x_t = (-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)'$,
  $\sigma^2 = 1, T = 21$.

Let $\hat{\beta}^{(i)}, i = 1, \cdots, 6$ be the least square estimate of β for the i-th case. Answer the following questions.

1. Write down the general formula of $\hat{\beta}$ in term of x_t, y_t as in eq. (1). (2)
2. Rank the precision of $\hat{\beta}^{(i)}, i = 1, \cdots, 6$ in each case. Order from the most precise estimate to the least precise one. Explain your results. No partial credit is given to answer without explanation. (10)
3. Let $|x_t| \leq 10$, $t = 1, \cdots, 20$, and the values of x_t is under your control. What values of x_t you should give to maximize the precision of $\hat{\beta}$?. Explain your answer. Note that ϵ_t, $t = 1, \cdots, 20$ is drawn randomly from $iid\, N(0, 1)$. (4)

8 Normal Random variables (15%)

Let (Y_1, Y_2) be bivariate normal variables with mean (1,-1) and covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

1. Write down the joint density function of (Y_1, Y_2).
2. Find the conditional distribution of Y_1 given that $Y_2 = 1$.
3. Prove that $E(Y_2) = E[E(Y_2|Y_1)]$.

9 Tchebysheff's theorem (10 %)

Let Y be a random variable mean 1, variance 4 and unknown distribution. What is your best guess of the probability $P(-3 < Y < 5)$. Explain your answer.