Matrix Algebra

October 4, 2007

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1 Terminology

A vector is an ordered set of numbers arranged either in a row or a column.

$$a = [a_1, \cdots, a_n]'$$

a is a column vector and $b = a'$ is a row vector. A **matrix** is a rectangular of numbers

$$A = [a_{ik}] = [A]_{ik} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}$$

- symmetric: $a_{ik} = a_{ki}$
- diagonal matrix: $a_{ij} = 0 \ \forall \ j \neq j$
- scalar matrix: $a_{ii} = c, a_{ij} = 0 \ \forall \ i \neq j$
- identity matrix: $a_{ii} = 1 \ \forall \ i, a_{ij} = 0 \ \forall i \neq j$
- triangular matrix: upper (lower) triangular matrix: $a_{i,j} = 0 \ \forall \ i > j (i < j)$

2 Algebraic manipulation of matrices

- $A = B$ if $a_{ik} = b_{ik}, \forall i, k$
- transpose $B = A' \leftrightarrow b_{ik} = a_{ki}$
  $(A')' = A$
- Matrix addition: $C = A + B = [a_{ik} + b_{ik}], A - B = [a_{ik} - b_{ik}]$
- Vector multiplication: $a'b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

- Matrix multiplication and scalar multiplication: $C = AB \Rightarrow c_{ik} = a_i' b_k$. (product of i-th row of $\boldsymbol{A}$ and j-th column of B. $cA = [ca_{ik}], c = A\boldsymbol{b} = b_1\boldsymbol{a}_1 + b_2\boldsymbol{a}_2 + \cdots + b_k\boldsymbol{a}_k$.
  $C = AB \Leftrightarrow c_k = Ab_k,\ AI = A, IA = A$.

- associate law: $(AB)C = A(BC)$
  distributed law: $A(B + C) = AB + AC$
  transpose of a product: $(AB)' = B'A'$ transpose of extended product: $(ABC)' = C'B'A'$

- sum of values:

$$
\begin{aligned}
[X'X]_{kl} &= [x_k' x_l] \\
X'X &= \sum_{i=1}^{n} x_i x_i'
\end{aligned}
$$

- idempotent matrix: $M^2 = MM = M$ example: $M = I - \frac{1}{n}ii'$

3 Geometry of matrices

1. vector spaces: any set of vectors that is closed under scalar multiplication and addition.
2. basic vectors: any vector in the vector space can be written as a linear combination of the set of vectors.
3. linear independence:
4. spanning vectors
5. column space
6. column rank: dimension of the vector space that is spanned by its column vectors.
7. rank(A)=rank(A') $\leq$ min(numbers of rows, number of columns)
   rank$(AB) \leq$ min(rank(A),rank(B)), rank(A)=rank(AA')=rank($A'A$)

4 Determinant of a matrix

$|A| = \sum_{i=1}^{k} a_{ik}(-1)^{i+k}|A_{ik}|$

5 Least squares problem

$$\begin{aligned} y &= Xb + e \\ e &\perp Xb \\ X'y &= X'Xb \end{aligned}$$

6 Solution of a system of linear equations

1. homogeneous equations: $Ax = 0$
2. nonhomogeneous systems of equations: $Ax = b$
3. inverse matrices: $AA^{-1} = I, A^{-1}A = I$
   $a_{ik} = \frac{C_{ki}}{|A|}$ where C_{ki} is the cofactor of A. $|A^{-1}| = \frac{1}{|A|},\ (AB)^{-1} = B^{-1}A^{-1}$
4. nonsingular matrix: its inverse exists.
5. solving the least squares problem
   $X'y = X'Xb \Rightarrow b = (X'X)^{-1}(X'y)$

7 Partitioned matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

1. addition and multiplication of partitioned matrices
2. determinant of partitioned matrices

$$|\left|\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|$$

3. inverse of partitioned matrices:
4. deviation from means

8 Kronecker products

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1k}B \\ a_{21}B & a_{22}B & \cdots & a_{2k}B \\ & & \cdots & \\ a_{n1}B & a_{n2}B & \cdots & a_{nk}B \end{bmatrix}$$

$$\begin{aligned} (A \otimes B)^{-1} &= (A^{-1} \otimes B^{-1}) \\ |A \otimes B| &= |A|^n |B|^m \text{ where } A = M \times M, B = n \times n \\ (A \otimes B)' &= A' \otimes B' \\ tr(A \otimes B) &= tr(A)tr(B) \\ (A \otimes B)(C \otimes D) = AC \otimes BD & \end{aligned}$$

9 Characteristic roots and vectors

$$\begin{aligned} Ac &= \lambda c \\ |A - \lambda I| &= 0 \end{aligned}$$

10 Diagonalization and spectral decomposition of a matrix

$$\begin{aligned} AC &= CA \\ C'C &= I \\ C'AC &= C'C\Lambda = I\Lambda = \Lambda \\ A &= C\Lambda C' = \sum_{i=1}^{k} \lambda_k c_k c_k' \end{aligned}$$

11 Rank of a matrix

the rank of a matrix A equals the number of nonzero characteristic roots in $A'A$.

12 condition number of a matrix

$$\gamma = \left[\frac{\text{maximum root}}{\text{minimum root}} \right]^{1/2}$$

13 Trace of a matrix

$tr(A) = \sum_{k=1}^{K} a_{kk}$

14 Determinant of a matrix

$$
\begin{aligned}
C'AC &= \Lambda \\
|C'AC| &= |\Lambda|
\end{aligned}
$$

15 Power of a matrix

$$
\begin{aligned}
AA &= A^2 = (C\Lambda C')(C\Lambda C') \\
&= C\Lambda^2 C' \\
A^{-1} &= (C\Lambda C')^{-1} = (C')^{-1}\Lambda^{-1}C^{-1} = C\Lambda^{-1}C'
\end{aligned}
$$

16 Idempotent matrices

$AA = A$

17 Factoring a matrix

- Spectral decomposition

$$
\begin{aligned}
P &= A^{-1/2}C' \\
P'P &= (C')'(\Lambda^{-1/2})'\Lambda^{-1/2}C' = C\Lambda^{-1}C'
\end{aligned}
$$

- Cholesky factorization
  $A = LU\, L' = U, A^{-1} = U^{-1}L^{-1}$

- $QR : X = QR, Q'Q = I, R$ is upper triangular.

18 generalized inverse of a matrix

$$
\begin{aligned}
AA^{+}A &= A \\
A^{+}AA^{+} &= A^{+} \\
A^{+}A \;\; &is \;\; \text{symmetric} \\
AA^{+} \;\; &is \;\; \text{symmetric}
\end{aligned}
$$

19 Quadratic forms and definite matrices

$q = x'Ax$

20 Calculus and matrix algebra

difference
optimization
constrained optimization transformations