2.4 Stochastic Calculus, the Ito Formula

In this section we want to develop some basic principles of "Stochastic Calculus". More precisely we want to formulate a version of the Fundamental Theorem of Calculus for stochastic processes.

Let us first recall the Fundamental Theorem of Calculus and its proof.

Theorem 2.4.1 *(The Fundamental Theorem of Calculus).*
Assume $f\colon\ [0,T] \to \mathbb{R}$ *is continuously differentiable.*
Then $f(T) - f(0) = \int\limits_0^T f'(t)dt.$

Proof. Let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0,T]$, $(0 = t_0 < t_1, \dots, t_n = T)$. Then

$$\begin{aligned} f(T) - f(0) &= \sum_{i=1}^n f(t_i) - f(t_{i-1}) \\ &= \sum_{i=1}^n \Delta t_i \frac{f(t_i) - f(t_{i-1})}{\Delta t_i} \\ &[\Delta t_i = t_i - t_{i-1}] \\ &= \sum_{i=1}^n \Delta t_i f'(t_i^*) \\ &[\text{with } t_i^* \in [t_{i-1}, t_i] \text{ chosen by the Mean Value Theorem}]. \end{aligned}$$

From the definition of Riemann integrals we deduce on the other hand that

$$\int\limits_0^T g(t)dt = \lim_{\|P\|\to 0} \sum_{i=1}^n \Delta t_i g(t_i^*) \quad [\text{with } t_i^* \in [t_{i-1}, t_i] \text{ arbitrary}].$$

Thus, we get

$$f(T) - f(0) = \lim_{\|P\|\to 0} \sum_{i=1}^n \Delta t_i f'(t_i^*) = \int\limits_0^T f'(t)dt. \qquad \square$$

We are given now a function $g\colon \mathbb{R} \to \mathbb{R}$, and a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{0 \le s < \infty}, \mathbb{P})$ and want to write

$$g(B_T) - g(B_0)$$

as a stochastic integral. But in this case we encounter some differences to the deterministic case.

Note that $g(B_t)$ is a random variable on Ω i.e. $g(B_t)\colon \Omega \ni \omega \longmapsto g(B_t(\omega))$. Assuming g being twice differentiable and bounded we obtain for a partition $P = \{t_0, t_1, \dots, t_n\}$ of $[0,T]$ from Taylor's expansion that

$$\begin{aligned} g(B_T) - g(0) &= \sum_{i=1}^{n} g(B_{t_i}) - g(B_{t_{i-1}}) \\ &= \sum_{i=1}^{n} g'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) + \sum_{i=1}^{n} \frac{1}{2} g''(\xi_i)(B_{t_i} - B_{t_{i-1}})^2, \end{aligned}$$

with ξ_i being an appropriately chosen random variable assuming its values between $B_{t_{i-1}}$ and B_{t_i}. If we let $\|P\| \to 0$ then $\sum\limits_{i=1}^{n} g'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})$ converges by Proposition 2.3.8 in L_2 to

$$\int_0^T g'(B_t) dB_t.$$

The problem is now the following: contrary to the deterministic case $\left(\text{i.e. } \sum\limits_{i=1}^{n} (t_i - t_{i-1})^2 g''(t_i^*)\right)$

$$\sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 g''(B_{t_i})$$

does in general not converge in L_2 to zero. Indeed, by Theorem 2.2.5 it follows that

$$\sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \longrightarrow T \quad \text{in} \quad L_2.$$

Thus, we will have supplementary terms in the stochastic version of the Fundamental Theorem of Calculus.

In order to simplify as much as possible the following analysis, let us assume for the moment that $g\colon \mathbb{R} \to \mathbb{R}$ is three times continuously differentiable and has a bounded first, second and third derivative, say

$$\sup_{x\in\mathbb{R}} \max\{|g'(x)|, |g''(x)|, |g'''(x)|\} = c < \infty.$$

This will make the derivations easier and more transparent. Later we will state more general results.

Using Taylor's expansion up to the third term we can write

$$\begin{aligned} g(B_T) - g(0) &= \sum_{i=1}^{n} g(B_{t_i}) - g(B_{t_{i-1}}) \\ &= \sum_{i=1}^{n} g'(B_{t_{i-1}})[B_{t_i} - B_{t_{i-1}}] && \text{(I)} \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} g''(B_{t_{i-1}})[B_{t_i} - B_{t_{i-1}}]^2 && \text{(II)} \\ &\quad + \frac{1}{6}\sum_{i=1}^{n} g'''(\xi_i)(B_{t_i} - B_{t_{i-1}})^3 && \text{(III)} \end{aligned} \tag{2.15}$$

where $P = \{t_0, t_1, \dots, t_n\}$ is a partition of $[0,T]$ and the ξ_i's are appropriately chosen random variables between $B_{t_{i-1}}$ and B_{t_i}.

We consider a sequence of partitions of $[0,T]$ $(P^{(n)})_{n\in\mathbb{N}}$, $P^{(n)} = (t_0^{(n)}, t_1^{(n)}, \dots t_n^{(n)})$, with $\|P^{(n)}\| \to 0$, and analyse what happens to the terms I, II, and III if n tends to ∞.

First we note that by Proposition 2.2.6 in Section 2.2 it follows that

$$\mathbb{E}\Big(\big(\sum_{i=1}^{n} |g''(\xi_i)||B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}|^3\big)^2\Big) \le c^2\mathbb{E}\Big(\big(\sum_{i=1}^{n} |B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}|^3\big)^2\Big) \to 0, \text{ if } n \to \infty. \tag{2.16}$$

This means that the third term (III) in (2.15) vanishes.

Secondly, it follows from Proposition 2.3.8 of Section 2.3 that

$$\mathrm{L}_2 - \lim_{n\to\infty} \sum_{i=1}^{n} g'(B_{t_{i-1}^{(n)}})[B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}] = \int_0^T g'(B_t)\,dB_t, \tag{2.17}$$

The following Lemma will handle the term (II) in (2.15). It can be seen as a generalization of Theorem 2.2.5 in Section 2.2.

Lemma 2.4.2 . *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded.*

Then

$$L_2 - \lim_{n\to\infty} \sum_{i=0}^{n-1} f(B_{t^{(n)}_{i-1}})(B_{t^{(n)}_i} - B_{t^{(n)}_{i-1}})^2 = \int_0^T f(B_t)dt.$$

Note that the left side of the equation has to be understood pointwise in the Riemann sense: for $\omega \in \Omega$ $\left(\int_0^T f(B_t)dt\right)(\omega)$ is the integral of the continuous function $[0,T] \ni t \mapsto f(B_t(\omega))$.

Proof of Lemma 2.4.2. Assume $|f(x)| \leq c$, whenever $x \in \mathbb{R}$, for some $c > 0$. For $\in \mathbb{N}$ define the following two random variables.

$$\begin{aligned} Y^{(n)} &= \sum_{i=0}^{n-1} f(B_{t^{(n)}_i})(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 \text{ and} \\ Z^{(n)} &= \sum_{i=0}^{n-1} f(B_{t^{(n)}_i})(t^{(n)}_{i+1} - t^{(n)}_i) \end{aligned}$$

Since for $i < j$ we deduce that

$$\begin{aligned} &\mathbb{E}\Big(f(B_{t^{(n)}_i})\big[(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 - (t^{(n)}_{i+1} - t^{(n)}_i)\big] f(B_{t^{(n)}_j})\big[(B_{t^{(n)}_{j+1}} - B_{t^{(n)}_j})^2 - (t^{(n)}_{j+1} - t^{(n)}_j)\big]\Big) \\ &\quad = \mathbb{E}\Big(f(B_{t^{(n)}_i})\big[(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 - (t^{(n)}_{i+1} - t^{(n)}_i)\big] f(B_{t^{(n)}_j})\mathbb{E}\Big(\big[(B_{t^{(n)}_{j+1}} - B_{t^{(n)}_j})^2 - (t^{(n)}_{j+1} - t^{(n)}_j)\big]\Big|\mathcal{F}_{t^{(n)}_j}\Big)\Big) = 0, \end{aligned}$$

it follows that

$$\begin{aligned} \mathbb{E}\Big(\big(Y^{(n)} - Z^{(n)}\big)^2\Big) &= \mathbb{E}\Big(\big(\sum_{i=0}^{n-1} f(B_{t^{(n)}_i})[(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 - (t^{(n)}_{i+1} - t^{(n)}_i)]\big)^2\Big) \\ &= \mathbb{E}\Big(\sum_{i=0}^{n-1} f^2(B_{t^{(n)}_i})[(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 - (t^{(n)}_{i+1} - t^{(n)}_i)]^2\Big) \\ &\leq c^2\mathbb{E}\Big(\sum_{i=0}^{n-1}[(B_{t^{(n)}_{i+1}} - B_{t^{(n)}_i})^2 - (t^{(n)}_{i+1} - t^{(n)}_i)]^2\Big) \\ &\to 0, \text{ as shown in the proof of Theorem 2.2.5.} \end{aligned}$$

Secondly, we note that by the definition of Riemann integrals, it follows that for each $\omega \in \Omega$ $Z^{(n)}(\omega)$ converges to $\int_0^T f(B_t(\omega))dt$. Since $|Z^{(n)}(\omega)| \leq cT$ for all $\omega \in \Omega$ we deduce from the Theorem of Majorized Convergence that $Z^{(n)}$ converges in L_2 to $\int_0^T f(B_t(\omega))dt$.

Thus by the triangle inequality

$$\|Y^{(n)} - \int_0^T f(B_t)dt\|_{\mathrm{L}_2} \leq \|Y^{(n)} - Z^{(n)}\|_{\mathrm{L}_2} + \|Z^{(n)} - \int_0^T f(B_t)dt\|_{\mathrm{L}_2} \to 0, \text{ for } n \to \infty. \qquad \square$$

Using now the equations (2.16) and (2.17) as well as the result of Lemma 2.4.2 we deduce from equation (2.16) that

$$g(B_T) - g(0) = \int_0^T g'(B_s)dB_s + \frac{1}{2}\int_0^T g''(B_s)ds$$

for functions $g\colon \mathbb{R} \to \mathbb{R}$ being 3 times continuously differentiable with bounded third and second derivatives.

Using now the more general stochastic integral as defined in Theorem 2.3.11 for elements of $\mathcal{H}_2^w([0,T])$ we deduce with a little more work but essentially the same ideas the following formula.

Theorem 2.4.3 *(Special Ito-formula).*
Assume $g(\cdot,\cdot) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is $(t,x) \mapsto g(t,x)$ is once continuously differentiable in t and twice continuously differentiable in x then

$$g(t,B_t) - g(0,B_0) = \int_0^t \frac{\partial g}{\partial s}(s,B_s)ds + \int_0^t \frac{\partial g}{\partial x}(s,B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s,B_s)ds$$

Remark. Ito's formula allows us to find $\int_0^t g(B_s)dB_s$ as follows.

Assume g is continuously differentiable, and let G be an antiderivative of g. Then the formula of Ito implies that

$$G(B_t) - G(0) = \int_0^t g(B_s)dB_s + \frac{1}{2}\int_0^t g'(B_s)ds.$$

Thus

$$\int_0^t g(B_s)dB_s = G(B_t) - G(0) - \frac{1}{2}\int_0^t g'(B_s)ds.$$

We now want to generalize the notion of stochastic integrals. Instead of integrating with respect to only the Brownian Motion we introduce integration with respect to "diffusion processes", a class of processes we will use to model stock prices.

Definition. Let (X_t), and (Y_t) be two processes so that their restriction to $[0,T]$ is in $\mathcal{H}_2^w([0,T])$ for any $T \geq 0$. Recall that this means that they are progressively measurable and

$$\mathbb{P}\Big(\Big\{\int_0^T X_u^2 du < \infty\Big\}\Big) = \mathbb{P}\Big(\Big\{\int_0^T Y_u^2 du < \infty\Big\}\Big) = 1.$$

Let Z_0 be $\mathcal{F}_0$-measurable.

The process Z_t with

$$Z_t = Z_0 + \int_0^t X_u du + \int_0^t Y_u dB_u. \tag{2.18}$$

is called a *diffusion process.*

We also write instead of (2.18)

$$dS_t = X_t dt + Y_t dB_t. \tag{2.19}$$

Remark. Formally the definition of Equation (2.19) is given by Equation (2.18). Nevertheless (2.19) has a more intuitive interpretation: Changes of Z_t over small time periods consist on the one hand of the deterministic "drift term" $X_t dt$ and on the other hand of the random "diffusion term" $Y_t dB_t$.

As already mentioned, diffusion processes will be our model for stock prices. Therefore we will have to define stochastic integration with respect to these processes.

Definition. A progressively measurable process (H_t) is called weakly square integrable with respect to a diffusion process (Z_t), were Z_t is given by

$$Z_t = Z_0 + \int_0^t X_u du + \int_0^t Y_u dB_u$$

if for all $T > 0$ it follows that

$$\mathbb{P}\Big(\Big\{\int_0^T (H_uX_u)^2du < \infty\Big\}\Big) = \mathbb{P}\Big(\Big\{\int_0^T (H_uY_u)^2du < \infty\Big\}\Big) = 1.$$

Note that this means that for all $T > 0$ the processes (H_uX_u) and (H_uY_u) are elements of $\mathcal{H}_2^w(0,T])$.

Therefore we can define the *stochastic integral of* (H_u) *with respect to* (Z_u) *on the interval* $[s,t]$ by

$$\int_s^t H_u dZ_u = \int_s^t H_uX_udu + \int_s^t H_uY_udB_u \tag{2.20}$$

Theorem 2.3.11 of Section 2.3 can be easily extended.

Proposition 2.4.4 . *Given a diffusion process* $dZ_t = X_tdt + Y_tdB_t$,

1) *If* $s < t$, α, β *are* $\mathcal{F}_s$*-measurable, and* H *and* G *are weakly square integrable with respect to* Z_t *then*

$$\int_s^t \alpha H_u + \beta G_u dZ_u = \alpha \int_s^t H_u dZ_u + \beta \int_s^t G_u dZ_u.$$

2) *If* $s < r < t$ *and* H *is square integrable with respect to* Z_t *then*

$$\int_s^t H_u dZ_u = \int_s^r H_u dZ_u + \int_r^t H_u dZ_u.$$

Remark. On one hand we defined in Equation (2.13) of Section 2.3 the stochastic integral of an elemantary adapted process with respect to a general adapted process. We have to verify that in the case of H being an elemntary process and Z being a diffusion process the definition in (2.13) coincides with the definition given in Equation (2.20). Secondly, the definition of Equation (2.13) was derived from our intuition on how gains and losses should be defined for a strategy H and we have to make sure that Equation (2.20) still coincides with that intuition.

Thus let $H_u = \sum_{i=0}^{n-1} h_i 1_{[t_i,t_{i+1})}$ be an elementary adapted process being square integrable with respect to Z. We observe that

$$
\begin{aligned}
\int_s^t H_u dZ_t &= \int_s^t H_u X_u du + \int_s^t H_u Y_u dB_u \\
&\text{[in the sense of Equation (2.20)]} \\
&= \sum_{i=0}^{n-1} \int_{(t_i \vee s)\wedge t}^{(t_{i+1}\vee s)\wedge t} h_i X_u du + \int_{(t_i \vee s)\wedge t}^{(t_{i+1}\vee s)\wedge t} h_i Y_u dB_u \\
&= \sum_{i=0}^{n-1} h_i \int_{(t_i \vee s)\wedge t}^{(t_{i+1}\vee s)\wedge t} X_u du + h_i \int_{(t_i \vee s)\wedge t}^{(t_{i+1}\vee s)\wedge t} Y_u dB_u \\
&= \sum_{i=0}^{n-1} h_i (Z_{(t_{i+1}\vee s)\wedge t} - Z_{(t_i\vee s)\wedge t}) \\
&= \int_s^t H_u dZ_t \\
&\text{[in the sense of Equation (2.13)]}
\end{aligned}
$$

Now we can state the Ito Formula for diffusion processes.

Theorem 2.4.5 *(General Ito formula).*
Assume Z_t is a diffusion process $dZ_t = X_t\ dt + Y_t\ dB_t$, $(X_t)_{t\geq 0}$, and $g\colon\ [0,\infty)\times\mathbb{R} \to \mathbb{R} \quad (t,x) \mapsto g(t,x)$ is continuously differentiable in t and twice continuously differentiable in x then

$$
g(T, Z_T) - g(0, Z_0) = \int_0^T \frac{\partial g}{\partial t}(t, Z_t)dt + \int_0^T \frac{\partial g}{\partial x}(t, Z_t)dZ_t + \frac{1}{2}\int_0^T \frac{\partial^2 g}{\partial x^2}(t, Z_t)Y_s^2\ ds
$$

with

$$
\int_0^T \frac{\partial g}{\partial x}(t, Z_t)dZ_t = \int_0^T \frac{\partial g}{\partial x}(t, Z_t)X_t\ dt + \int_0^t \frac{\partial g}{\partial x}(t, Z_t)Y_t\ dB_t.
$$

Remark: Here is an informal way to remember the laws of stochastic calculus:

Now using the second Taylor expansion in differential form for $dZ_t = X_t\ dt + Y_t\ dB_t$

$$
\begin{aligned}
d(g(t,Z_t)) = \frac{\partial}{\partial t} g(t,Z_t)dt + &\underbrace{\frac{\partial}{\partial x} g(t,Z_t)dZ_t}_{=\frac{\partial}{\partial x}g(t,Z_t)X_t dt + \frac{\partial}{\partial x}g(t,Z_t)Y_t dB_t} \\
&+ \underbrace{\frac{1}{2}\frac{\partial^2}{\partial t^2} g(t,Z_t)d^2 t}_{=0} + \underbrace{\frac{\partial^2}{\partial t \partial x} g(t,Z_t)dt\ dZ_t}_{=0} \\
&+ \underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2} g(t,Z_t)d^2 Z_t}_{=\frac{1}{2}\frac{\partial^2}{\partial x^2}g(t,Z_t)Y_t^2\ dt} .
\end{aligned}
$$

Law:	precise meaning:
$(dt)^2 = 0$	$\lim\limits_{\substack{\|P_n\|\to 0 \\ s=t_0^{(n)}<t_1^{(n)}<\cdots<t_n^{(n)}=t}} \sum_{i=1}^{n} (t_i^{(n)} - t_{i-1}^{(n)})^2 = 0$
$dt\ dB_t = 0$	$\lim\limits_{\substack{\|P_n\|\to 0 \\ s=t_0^{(n)}<t_1^{(n)}<\cdots<t_n^{(n)}=t}} \sum_{i=1}^{n} \lvert t_i^{(n)} - t_{i-1}^{(n)}\rvert \, \lvert B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}\rvert = 0$
$(dB_t)^2 = dt$	$\lim\limits_{\substack{\|P_n\|\to 0 \\ s=t_0^{(n)}<t_1^{(n)}<\cdots<t_n^{(n)}=t}} \sum_{i=1}^{n} (B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}})^2 = t - s.$

After this rather abstract and technical introduction of the basics of stochastic processes we are in the position to introduce a model for the stock price (S_t). We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$.

$(S_t)_{t\geq 0}$ is then a stochastic process satisfying a "stochastic differential equation" of the following form

$$dS_t(\omega) = \mu(t, \omega, S_t(\omega))S_t(\omega)dt + \nu(t, \omega, S_t(\omega))S_t(\omega)dB_t.$$

Or, in integral form is equation can be written as:

$$S_t(\omega) - S_0 = \int_0^t \mu(s, \omega, S_s(\omega))S_s(\omega)ds + \int_0^t \nu(s, w, S_s(\omega))S_s(\omega)dB_s.$$

where $(\mu(t, \cdot, S_t(\cdot))_{t\geq 0}$ (the drift) and $(\nu(t, \cdot, S_t(\cdot))_{t\geq 0}$ (volatility) are adapted piecewise continuous processes whenever $(S_t(\cdot))_{t\geq 0}$ has these properties.

Note, that we haven't given an "explicit model" for the stock price yet. Since "S_t" appears on both sides of the equation, it is only an "implicit" description. The following theorem on stochastic differential equations gives conditions which insure the uniqueness and existence of a solution:

Theorem 2.4.6 . *Assume $(\mu(t,\cdot,X_t(\cdot)))_{t\geq 0}$ and $\nu(t,\cdot,(X_t(\cdot))_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$ adapted piecewise continuous, locally square integrable, whenever $(X_t)_{t\geq 0}$ has these properties. Also assume the following "Lipschitz condition" in the third argument, i.e. $\exists K>0$ so that for all $t\geq 0$, $\omega\in\Omega$*

$$|x\mu(t,\omega,x)-y\mu(t,\omega,y)|\leq K|x-y|$$
$$|x\nu(t,\omega,x)-y\nu(t,\omega,y)|\leq K|x-y|.$$

Then the stochastic differential equation

$$dX_t=\mu(t,\cdot,X_t)X_t\ dt+\nu(t,\cdot,X_t)X_t\ dB_t$$

has a unique solution $(X_t)_{t\geq 0}$

$$\left[i.e.\ X_t-X_0=\int_0^t \mu(s,\cdot,X_s)X_s\ ds+\int_0^t \nu(s,\cdot,X_s)X_s\ dB_s\right].$$

Example: Let us assume the drift μ and the volatility ν being constant. What is the solution of the following SDE ?

$$(*)\qquad dS_t=\mu S_t\ dt+\nu S_t\ dB_t$$

We apply Ito's formula to $g(t,x)=\ln x$

$$d\ln(S_t)=\frac{dS_t}{S_t}-\frac{1}{2}\frac{1}{S_t^2}\nu^2S_t^2\ dt=\frac{dS_t}{S_t}-\frac{\nu^2}{2}dt$$

Thus by solving for $\frac{dS_t}{S_t}$ and integrating we derive that

$$\int_0^t \frac{dS_s}{S_s}=\ln(S_t)-\ln(S_0)+\frac{\nu^2}{2}t.$$

On the other hand $(*)$ implies that

$$\int_0^t \frac{dS_t}{S_s}=\mu t+\nu B_t.$$

which implies that

$$\ln(S_t) - \ln(S_0) + \frac{\nu^2}{2}t = \mu t + \nu B_t$$

or

$$S_t = S_0 \cdot e^{\nu B_t + (\mu - \frac{\nu^2}{2})t}.$$