Teaching Notes on Impulse Response Function and Structural VAR

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1 Introduction

Structural VAR embeds economic theory within time series models, providing a convenient and powerful framework for policy analysis. Impulse response function (IRF) tracks the impact of any variable on others in the system. It is an essential tool in empirical causal analysis and policy effectiveness analysis. This note reviews important concepts related to impulse response function and structural VAR.

2 Impulse response function

Let Y_t be a k-dimensional vector series generated by

$$\begin{aligned} Y_t &= A_1 Y_{t-1} + \cdots + A_p Y_{t-p} + U_t \\ &= \Phi(B) U_t = \sum_{i=0}^{\infty} \Phi_i U_{t-i} \qquad (1) \\ I &= (I - A_1 B - A_2 B - \cdots - A_p B^p)\Phi(B) \qquad (2) \end{aligned}$$

where $cov(U_t) = \Sigma, \Phi_i$ is the MA coefficients measuring the impulse response. More specifically, $\Phi_{jk,i}$ represents the response of variable j to an unit impulse in variable k occurring i-th period ago. IRF are used to evaluate the effectiveness of a policy change, say increasing rediscount rate.

As Σ is usually non-diagonal, it is impossible to shock one variable with other variables fixed. Some kind of transformation is needed. Cholesky decomposition is the most popular one which we shall turn to now. Let P be a lower triangular matrix such that $\Sigma = PP'$. then eq. (1) can be rewritten as

$$Y_t = \sum_{i=0}^{\infty} \Theta_i w_{t-i}$$

where $\Theta_i = \Phi_i P, w_t = P^{-1} U_t$, and $E(w_t w_t') = I$. Let D be a diagonal matrix with same diagonals with P and $W = PD^{-1}, \Lambda = DD'$. After some manipulations, we obtain

$$Y_t = B_0 Y_t + B_1 Y_{t-1} + \cdots + B_p Y_{t-p} + V_t$$

where $B_0 = I_k - W^{-1}, W = PD^{-1}, B_i = W^{-1} A_i$. Obviously, B_0 is a lower triangular matrix with 0 diagonals. In other words, Cholesky decomposition imposes a recursive causal structure from the top variables to the bottom variables but not the other way around.

Useful remarks

1. For a K-dimensional stationary VAR(p) process,

$$\phi_{jk,i} = 0, \quad \text{for } j \neq k,\ i = 1, 2, \cdots$$

   is equivalent to

$$\phi_{jk,i} = 0 \quad \text{for } i = 1, \cdots, p(K-1)$$

   .

   In other words, if the first $pK - p$ responses of variable j to an impulse in variable k is zero, then all the following responses are all zero. (Lutkepohl Proposition 2.4).

2. Variable k does not cause variable j if and only if $\Phi_{jk,i} = 0, i = 1, 2, \cdots$.

Critiques of IRF

1. Sensitive to variables ordering.
   Generalized impulse response function by Pesaran offers a partial solution and Granger and Swanson (1997) proposed a different but more promising one.

2. Omitting important variables may lead to major distortions in IRF and make the empirical results worthless. However, its impact on forecasting could small. *Why?*

3 Generalized impulse response function

To circumstance the problem of ordering dependence of IRF, Pesaran and Shin (1998) proposed the GIRF.

$$\begin{aligned}
X_t &= \sum_{i=1}^{p} A_i X_{t-i} + U_t \\
&= \sum_{i=0}^{\infty} \Phi_i U_{t-i} \\
\Phi_i &= A_1 \Phi_{i-1} + A_2 \Phi_{i-2} + \cdots + A_p \Phi_{i-p}, \quad i = 1, 2, \cdots
\end{aligned}$$

where $E(U_t U_t') = \Sigma$

Cholesky decomposition of Σ, $PP' = \Sigma$ so that

$$X_t = \sum_{i=0}^{\infty}(A_i P)(P^{-1}U_{t-i})$$

IRF is

$$\Psi_j^o(n) = \Phi_n P e_j, n = 0, 1, 2, \cdots$$

where e_j is an $m \times 1$ selection vector with unity as its j-th element and zeros elsewhere.

GIRF is defined as :

$$GIRF_x(n, \delta_j, \Omega_{t-1}) = E(X_{t+n}|u_{jt} = \delta_j, \Omega_{t-1}) - E(X_{t+n}|\Omega_{t-1})$$

Assume normal distribution for U_t

$$E(U_t|U_{jt} = \delta_j) = (\sigma_{1j}, \sigma_{2j}, \cdots, \sigma_{mj})'\sigma_{jj}'\delta_j = \Sigma U_j e_j \sigma_{jj}^{-1}\delta_j$$

Unscaled GIRF is:

$$(\frac{\Phi_n \Sigma e_j}{\sqrt{\sigma_{jj}}})(\frac{\delta_j}{\sqrt{\sigma_{jj}}}), n = 0, 1, 2, \cdots$$

Scaled GIRF by setting $\delta_j = \sqrt{jj}$,

$$\Psi_j^g(n) = \sigma_{jj}^{-1/2}\Phi_n \Sigma U_j, n = 0, 1, 2, \cdots$$

Forecast error decomposition:

$$\theta_{ij}^o = \frac{\sum_{l=0}^{n}(U_l'\Phi_j P U_j)^2}{\sum_{l=0}^{n}(U_i'\Phi_l \Sigma A_l' U_i)}; \quad \theta_{ij}^g = \frac{\sigma_{ii}^{-1}\sum_{l=0}^{n}(U_i'\Phi_l \Sigma U_j)^2}{\sum_{l=0}^{n} U_i'\Phi_l \Sigma \Phi_l' U_i}, i, j = 1, \cdots, m$$

Note that $\sum_{j=1}^{m}\theta_{jj}^o(n) = 1, \sum_{j=1}^{m}\theta_{jj}^g(n) \neq 1.$

Comparing GIRF and IRF

1. In stead of controlling the impact of correlation among residuals, GIRF follows the idea of nonlinear impulse response function and compute the mean impulse response function. When one variable is shocked, other variables also vary as is implied by the covariance. GIRF computes the mean by integrating out all other shocks.

2. When Σ is diagonal, GIRF is the same as IRF.

3. GIRF is unaffected by ordering of variables

4. The generalized impulse response of the effect of an unit shock to j-th equation is the same as that of an orthogonal impulse response but different for other shocks. To be specific,

$$\begin{array}{rcl} \Phi_1^g(n) & = & \Phi_1^o(n) \\ \Phi_j^g(n) & \neq & \Phi_j^o(n), \quad j = 2, 3, \cdots, m \end{array}$$

   Thus the GIRF can be easily computed by usual IRF with each variable as leading one.

5. The formula of GIRF is derived under the assumption of multivariate normality that might not be true for some empirical applications.

4 Unit Root, Cointegration and IRF

1. If there exists unit roots and/or cointegration, then estimated IRF is inconsistent at long horizons in unrestricted VARs. Error correction model produces consistent IRF and optimal predictions.

2. IRF estimates based upon ECM is consistent.

3. Proper procedures for computing IRF for a cointegrated system are:

   (a) Determine the cointegration rank by LR test;

   (b) Estimate the ECM model: $\Delta Y_t = \alpha\beta' Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \Phi D_t + U_t$;

   (c) Converted the ECM back to VAR model;

   (d) Use the resulting VAR model to perform IRF.

5 Structural VAR

This part is taken from the VAR.SRC written by Norman Morin (nmorinfrb.gov).
REDUCED FORM

$$Y_t = A_1 * Y_{t-1} + \cdots + A_p Y_{t-p} + W * Z(t) + c + d * t + U_t$$

where Y denotes vector of endogenous variables of interest, X vector of exogenous variables, U is vector of residuals and $EU_tU_t' = \Sigma$. The innovations can be written terms of uncorrelated error terms

$$\begin{aligned} U_t &= G * U_t + E_t \\ E(E_tE_t') &= D \end{aligned}$$

where D is a diagonal matrix whose diagonals are the variances of E and G has zeroes on the diagonals. Now, let $B * U_t = E_t$ or $A * E_t = U_t$ where $B = I - G$, and $A = B^{-1}$, where B and A have unit diagonals Thus,

$$\begin{aligned} B * \Sigma * B' &= D = E(E_tE_t') \\ A * D * A' &= \Sigma = E(U_tU_t') \end{aligned}$$

This will yield the structural form based on the orthogonalization.

$$B * Y_t = B_1 * Y_{t-1} + \cdots + B_p * Y_{t-p} + F * X(t) + v + k * t + E_t$$

with $B_i = B * A_i, i = 1, \cdots, p, F = B * W, v = B * c$, and $k = B * d$.

Given B and D, one can write a structural form vector moving average based on the reduced form matrices $A_1, \cdots, A_p$

$$\begin{aligned} Y_t &= U_t + C_1 * U_{t-1} + C_2 * U_{t-2} + \cdots \\ Y_t &= M_0 * E_t + M_1 * E_{t-1} + M_2 * E_{t-2} + \cdots \end{aligned}$$

The coefficient $(i, j)th$ element of M_k is the effect on variable i of a shock to j-th structural form innovation k periods ago.

The various choices of orthogonalizations for impulse responses place conditions on the structural form matrices B and D:

1. CHOLESKY:
   Factors Σ into $P * P'$ where P is lower triangular whose diagonals are the standard deviations of E. Thus, the first variable in the VAR is only affected contemporaneously by the shock to itself. The second variable in the VAR is affected contemporaneously by the shocks to the first variable and the shock to itself, and so on... $P = B^{-1}D^{1/2}$

2. BERNANKE-SIMS:
   Factors Σ into $B^{-1}DB^{-1'}$ where D is diagonal (with the variances of E), B has unit diagonals, but allows for the user to force certain $B(i, j) = 0$,

(not for $i = j$) and will test these restrictions. You are asked for the number of nonzero NONDIAGONAL free coefficients, and is then asked to input the row number and column number for each non-diagonal free coefficient (this is done by entering the row number and column number separated by a comma (preferred) or a space).

3. HARVEY-SARGAN:
   Factors Σ into $B^{-1}DB^{-1'}$ where the user can distribute unit coefficients and zeros among both B and D, but one or the other will have unit diagonals. The user chooses which matrix will contain unit diagonals, is then prompted for the number of NON-DIAGONAL free coefficients in B and in D, and is then asked to input the row number and column number for each non-diagonal free coefficient (this is done by entering the row number and column number separated by a comma (preferred) or a space).

4. IDENTITY:
   Assumes Σ is diagonal, i.e., the reduced form innovations are contemporaneously uncorrelated.

5. BLANCHARD-QUAH:
   Factors Σ into PP' where $P = C_1^{-1}G$ where G is the LT Cholesky decomposition of $C_1\Sigma_1'C_1'$ and C_1 is the sum of the ∞-order VMA coefficients from the Wold decomposition of the VAR. This yields impulse responses such that the 1st variable may have long run effects on all variables, the 2nd may have long run effects on all but the 1st, the 3rd on all but the 1st and 2nd, etc .... In the BQ article, shocks are assigned as ”supply” and ”demand” shocks, without reference to a variable ordering. Here, the shocks are labeled with the ordering of the variables in the VAR, but need not be given that interpretation. $B = P^{-1}$ and $D = I$.

NOTE that 2, 3, and 4 will test the restrictions. For 2 and 3, the number of free coefficients (restrictions) should be less than or equal to $p(p + 1)/2$, where p is number of variables, and there must be no zeros on the diagonals

Softwares

1. Impulse responses: Reduced form and structural form

   - VAR.SRC/RATS by Norman Morin
   - SVAR.SRC/RATS by Antonio Lanzarotti and Mario Seghelini

- VAR/View/Impulse/Eviews
- FinMetrics/Splus

2. Cointegration:
   - CATS/RATS
   - COINT2/GAUSS
   - VAR/Eviews
   - urca/R
   - FinMetrics/Splus

3. Impulse response under cointegration constraint:
   CATS,CATSIRFS/RATS

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