Time Series Analysis:
Exercise 1

due on March 23, 2010

1. Define and explain the following terms:

   - sigma field
   - random variable
   - measurable
   - Borel sets, Borel functions
   - probability space
   - stationary process
   - time series
   - strong mixing process
   - ergodic process
   - martingale process
   - Markov process
   - Brownian motion
   - ARMA process

2. Define ACF(autocorrelation function), PACF(partial autocorrelation function) and ESACF(extended autocorrelation function). Derive PACF for a stationary time series x_t. Find the ACF and PACF for the following series: processes:

   - $y_t = 0.8y_{t-1} + \epsilon_t$
   - $y_t = \epsilon_t + 0.8\epsilon_{t-1}$
   - $y_t = 0.8y_{t-1} + \epsilon_t + 0.8\epsilon_{t-1}$

3. Explain how to identify a time series model using ACF, PACF and ESACF.

4. Asymptotic theory
Let y_t be an AR(1): $y_t = \beta y_{t-1} + \epsilon_t, \beta = 0.8, \sigma^2 = 4$ as in previous question. The asymptotic theory suggests that the OLS estimate $\hat{\beta}$ of the AR coefficient follows an asymptotic normal distribution:

$$\hat{\beta} = \frac{\sum_{t=2}^{T}(y_t y_{t-1})}{\sum_{t=2}^{T} y_{t-1}^2} \tag{1}$$

$$\sqrt{T}(\hat{\beta} - \beta) \rightarrow N(0, \sigma_\beta^2) \tag{2}$$

What value should σ_β^2 be?

5. Let y_t be generated by

$$y_t = \beta x_t + \epsilon_t \tag{3}$$

$$\epsilon_t = \alpha \epsilon_{t-1} + \eta_t \tag{4}$$

where $|\alpha| < 1, x_t$ is non-stochastic regressor and η_t is white noise. Denote $\hat{\beta}$ be the least square estimate of β.

- Is $\hat{\beta}$ consistent estimate for β?
- Is $\hat{\beta}$ efficient estimate for β?

Prove or justify your answers.

6. Let y_t be generated by

$$y_t = \rho y_{t-1} + \epsilon_t, \quad \text{for } t > 0 \tag{5}$$

where $y_0 = k, \epsilon_t$ be iid with mean 0 and variance σ^2. Let $\hat{\rho}_T$ denote the least square estimate of ρ at time T. Derive the limiting distribution of $\sqrt{T}(\hat{\rho}_T - \rho_0)$ under the null hypothesis that $\rho = \rho_0$ where $|\rho_0| < 1$. How does σ^2 affect the limiting distribution? Explain your answer.

7. Let y_t be generated by an ARMA(p,q) as:

$$y_t = a_1 y_{t-1} + \ldots + a_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q}$$

(a) Explain why is the OLS of y_t on $y_{t-1}, \ldots, y_{t-p}$ is inconsistent?

(b) Let $p = q = 1$, derive the method of moment estimators for a_1, θ_1. Derive the MLE for $a_i, i = 1, \ldots, p, \theta_j, j = 1, \ldots, q$.

(c) Give an asymptotic theorem for the MLE above. Be specific about the covariance matrix