CAPM and betas

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Introduction

- Observation: stock is risky and should have higher return (risk premium)
- Expect a formulation like $r \sim \textit{variance}$ provided variance is used to measure risk
- Trouble is: Two different kinds of risk: market risk and idiosyncratic risk.
  The latter can be diversified away and is hence not priced.
  The former cannot be diversified away and should be priced.
  So, how do we proceed with the analysis?

- Markowitz(1959) argues that
  - investors would hold a mean-variance efficient portfolio, highest return for given level of variance or lowest variance with given exdpected return.
  - If investors have homogeneous expectations, and optimally hold mean-variance efficient portfolio, then in the absence of market frictions the market portfolio will be a mean-variance efficient portfolio.
- CAPM is a direct implication of the mean-variance efficiency market portfolio.

Markowitz Portfolio theory without a risk free asset

Let there be n risky assets with returns $r = (r_1, \cdots, r_n)'$, where $\mu = E(r) = E(r_1, \cdots, r_n)' = (\mu_1, \cdots, \mu_n)'$ and covariance matrix $\Omega = Cov(r)$ where

$$\Omega = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

Portfolio x is measured by the fractions of wealth hold for each asset, $x = (x_1, x_2, \cdots, x_n)$. Obviously, $x'\mathbf{i} = \sum_{j=1}^{n} x_j = 1$ where $\mathbf{i} = (1, 1, \cdots, 1)'$.

Portfolio p is the minimum-variance portfolios of all portfolios with mean return μ_p if x is the solution to the following constrained optimization:

$$min_x x'\Omega x$$

subject to

$$\begin{aligned} x'\mu &= \mu_p \\ x'\mathbf{i} &= 1 \end{aligned}$$

To solve the problem, form the Lagrangian function L and then differentiate with respect to x, δ_1, δ_2, respectively and solve for x.

$$L = x'\Omega x + \delta_1(\mu_p - x'\mu) + \delta_2(1 - x'\mathbf{i})$$

$$\begin{aligned} 2\Omega x - \delta_1\mu - \delta_2\mathbf{i} &= 0 && (1)\\ x'\mu &= \mu_p && (2)\\ x'\mathbf{i} &= 1 && (3) \end{aligned}$$

Multiplying Ω^{-1} to both sides of (1), we have

$$x = 1/2\Omega^{-1}(\delta_1\mu + \delta_2\mathbf{i}) \qquad (4)$$

Left-multiplying $\mathbf{i}'$ and μ' to both ends of (4) respectively and making use of (2,3), we have

$$\begin{aligned} 1 = \mathbf{i}'x &= 1/2(\mathbf{i}'\Omega^{-1}\mu)\delta_1 + 1/2(\mathbf{i}'\Omega^{-1}\mathbf{i})\delta_2\\ \mu_p = \mu'x &= 1/2(\mu'\Omega^{-1}\mu)\delta_1 + 1/2(\mu'\Omega^{-1}\mathbf{i})\delta_2 \end{aligned}$$

Let $A = \mathbf{i}'\Omega^{-1}\mu, B = \mu'\Omega^{-1}\mu, C = \mathbf{i}'\Omega^{-1}\mathbf{i}, D = BC - A^2$, then

$$\begin{pmatrix} 1 \\ \mu_p \end{pmatrix} = \begin{pmatrix} 1/2A & 1/2C \\ 1/2B & 1/2A \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

Solve for δ_1, δ_2 and we have

$$x_p = g + h\mu_p$$

where

$$\begin{aligned} g &= \frac{1}{D}[B(\Omega^{-1}\mathbf{i}) - A(\Omega^{-1}\mu)] \\ h &= \frac{1}{D}[C(\Omega^{-1}\mu) - A(\Omega^{-1}\mathbf{i})] \end{aligned}$$

Portfolio with a risk free asset

Let the return for the risk free asset be r_f. Then, the portfolio optimization amounts to solving

$$min_x x'\Omega x$$

subject to

$$x'\mu + (1 - x'\mathbf{i})r_f = \mu_p$$

The Lagrangian function L now becomes

$$L = x'\Omega x + \delta(\mu_p - x'\mu - (1 - x'\mathbf{i})r_f)$$

Differentiate L with respective to x and δ, we have

$$2\Omega x - \delta(\mu - r_f \mathbf{i}) = 0 \tag{5}$$

$$x'\mu + (1 - x'\mathbf{i})r_f = \mu_p \tag{6}$$

Multiplying Ω^{-1} to both sides of (5), we have

$$x = 1/2\Omega^{-1}\delta(\mu - r_f \mathbf{i})$$

Note that (6) is equivalent to

$$\mu_p - r_f = x'(\mu - r_f \mathbf{i}) \tag{7}$$

Left-multiplying $(\mu - r_f \mathbf{i})'$ to (5) and making use of (7), we obtain

$$\delta = \frac{\mu_p - r_f}{(\mu - r_f \mathbf{i})'\Omega^{-1}(\mu - r_f \mathbf{i})}$$

Hence,

$$x_p = \frac{(\mu_p - r_f)}{(\mu - r_f \mathbf{i})'\Omega^{-1}(\mu - r_f \mathbf{i})}\Omega^{-1}(\mu - r_f \mathbf{i})$$

Note that

$$x_p = c_p \bar{x}$$

where

$$c_p = \frac{(\mu_p - r_f)}{(\mu - r_f \mathbf{i})' \Omega^{-1} (\mu - r_f \mathbf{i})}$$

and

$$\bar{x} = \Omega^{-1} (\mu - r_f \mathbf{i})$$

which latter does not depends upon p.
With a risk free asset, all minimum-variance portfolios are a combination of a given risky asset with weights proportional to $\bar{x}$ and the risk free asset. The portfolio of risky free assets is called the tangency portfolio and has weight vector

$$x_q = \frac{1}{\mathbf{i}' \Omega^{-1} (\mu - r_f \mathbf{i})} \Omega^{-1} (\mu - r_f \mathbf{i})$$

Denote x_q as market portfolio x_m for beng consistent with notations in the textbook. Now, we can derive the CAPM easily from portfolio optimization solution above. Evaluate (5) at x_q so that $x_m' \mathbf{i} = 1$ and multiple x_m' to both sides to get

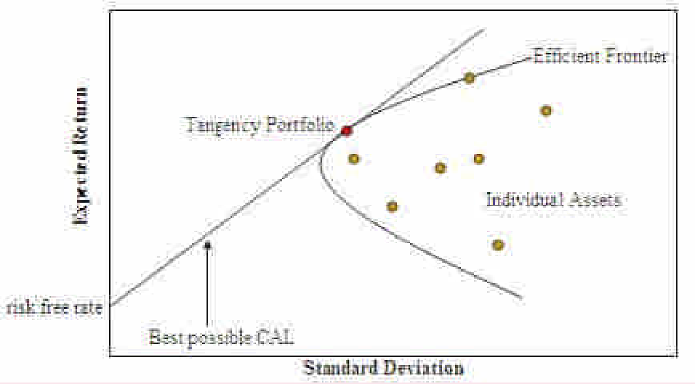
$$2x_m' \Omega x_m = \delta x_m' (\mu_m - r_f \mathbf{i}) = \delta x_m' (\mu_m - r_f)$$

where Hence,

$$\frac{1}{\delta} = \frac{\mu_m - r_f}{2x_m \Omega x_m} = \frac{\mu_m - r_f}{2\sigma_m^2}$$

substituting back to (6), we have

$$\mu - r_f \mathbf{i} = \frac{\mu_m - r_f}{\sigma_m^2} \Omega x_m$$


Efficient Frontier
Tangency Portfolio
Expected Return
Individual Assets
risk free rate
Best possible CAL
Standard Deviation

Note that $\sigma_{im} = x_1\sigma_{i1} + x_2\sigma_{i2} + \cdots + x_n\sigma_{in}$. Write out (11) in details:

$$\begin{bmatrix} \mu_1 - r_f \\ \mu_2 - r_f \\ \vdots \\ \mu_i - r_f \\ \vdots \\ \mu_n - r_f \end{bmatrix} = \frac{\mu_m - R_f}{\sigma_m^2} \begin{bmatrix} \sigma_{1m} \\ \sigma_{2m} \\ \vdots \\ \sigma_{im} \\ \vdots \\ \sigma_{nm} \end{bmatrix}$$

More specifically,

$$\mu_i - r_f = \frac{\sigma_{im}}{\sigma_m^2}(r_m - r_f) = \beta_i(r_m - r_f)$$

CAPM: Formula

$$\mu_i - r_f = \frac{\sigma_{im}}{\sigma_m^2}(r_m - r_f) = \beta_i(r_m - r_f)$$

CAPM:assumptions

1. Homogeneous expectations: All investors share the same beliefs about the distribution of returns.
2. Investors have the same one period time horizon
3. All investors can borrow or lend at the risk-free rate
4. No transactions costs
5. Investors are indifferent between capital gains and dividends
6. No inflation (no unanticipated inflation)
7. Large number of small investors (no one investors can impact the price on his or her own)
8. Markets are in equilibrium

CAPM: Important principles

1. The risk of a security depends on the portfolio to which it is added
2. In diversified portfolios, the dominant risk of a security is the covariance of its returns with other securities held in the portfolio
3. The risk of a security relative to the risks of other securities held in the portfolio can be calculated by dividing the covariance of then securitys returns with other securities held by the variance of returns on other securities held (the security's β)

CAPM:testing

Denote r_t as an $(n \times 1)$ vector of excess returns for n assets and r_{mt} as the market portfolio excess return. The excess return market model amounts to

$$\begin{aligned}
r_t &= \alpha + \beta r_{mt} + \epsilon_t \\
E(\epsilon_t) &= 0 \\
E(\epsilon_t \epsilon_t') &= \Sigma \\
E(r_{mt}) &= \mu_m \\
E(r_{mt} - \mu_m)^2 &= \sigma_m^2 \\
Cov[r_{mt}, \epsilon_t] &= 0
\end{aligned}$$

where β is the $(n \times 1)$ vectors of betas.

CAPM implies that $\alpha = 0$. Let $\hat{\alpha}, \hat{\Sigma}$ denote the least square or MLE of α, Σ respectively and the Wald test, J_0 and finite sample F-test are :

$$
\begin{aligned}
J_0 &= \hat{\alpha}'[Var[\hat{\alpha}]]^{-1}\hat{\alpha} \\
&= T[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}]^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim \chi^2(N) \\
J_1 &= \frac{T-N-1}{N}[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}]^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F(T-N-1, N)
\end{aligned}
$$