1.

(a)
$$e \equiv \frac{W_{Carnot}}{|Q_h|} = \frac{|Q_h|-|Q_c|}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|}$$

In a Carnot cycle, $\Delta E_{in} = 0$ (path $A \to B$)

$\therefore |Q_h| = |-W_{AB}|$ $W_{AB} \equiv$ work done between $A \to B$

$A \to B$, $|Q_h| = |-W_{AB}| = nRT_h \ln\left(\frac{V_B}{V_A}\right)$ absorb energy

$C \to D$, $|Q_c| = |-W_{CD}| = nRT_c \ln\left(\frac{V_C}{V_D}\right)$

$$\therefore \frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h} \frac{\ln\left(\frac{V_C}{V_D}\right)}{\ln\left(\frac{V_B}{V_A}\right)} \quad — ①$$

But $P_i V_i^{\gamma} = P_f V_f^{\gamma} \to T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$

$$\left.\begin{aligned} B \to C \quad & T_h V_B^{\gamma-1} = T_c V_C^{\gamma-1} \\ D \to A \quad & T_h V_A^{\gamma-1} = T_c V_D^{\gamma-1} \end{aligned}\right\} \Rightarrow \left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1} \quad \therefore \frac{V_B}{V_A} = \frac{V_C}{V_D} \quad — ②$$

From ① and ② $\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}$

Therefore $e_{Carnot} = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{T_c}{T_h}$ #

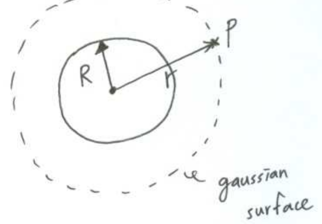
(b) For an engine to have perfect performance $e = 1$

$\Rightarrow |Q_c| = 0$ or $T_c \to 0$ #

(c) The perfect engine requires $T_c = 0$ not possible or $Q_c = 0$, that means it will not give up or waste any energy. This is not possible in reality.

2.

(1) A point outside the sphere : Pick a gaussian surface bigger than the surface of the sphere



$$\boxed{\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}}$$

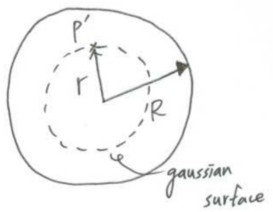
$$\Rightarrow E \oint dA = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}$$

$$= k_e \frac{Q}{r^2} \quad (r > R) \qquad \propto \frac{1}{r^2}$$

#

(2) A point inside the sphere : Pick a gaussian surface as shown



$$\boxed{\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{q_{in}}{\epsilon_0}, \quad q_{in} = \rho \cdot V$$

$$= \frac{Q}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3$$

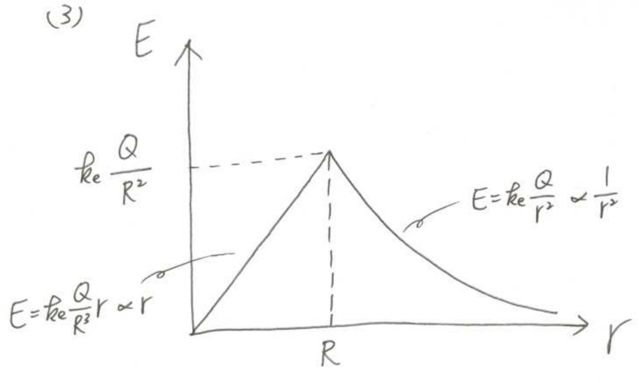
$$= Q \frac{r^3}{R^3}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r$$

$$= k_e \frac{Q}{R^3} r \quad (r < R) \qquad \propto r$$

#

(3)



#

3.

(1)

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{x^2+a^2}} = k_e \frac{1}{\sqrt{x^2+a^2}} \underbrace{\int dq}_{q} = k_e \frac{q}{\sqrt{x^2+a^2}}$$

#

(2)

$$E = \boxed{E_x = -\frac{dV}{dx}}$$

$$= -k_e q \frac{d}{dx}\left(\frac{1}{\sqrt{x^2+a^2}}\right)$$

$$= -k_e q \underbrace{\frac{d}{dx}(x^2+a^2)^{-\frac{1}{2}}}_{-\frac{1}{2}(x^2+a^2)^{-\frac{3}{2}} \cdot 2x} = k_e q \frac{x}{(x^2+a^2)^{\frac{3}{2}}}$$

#

4.

Energy stored in a charged capacitor :

The work needed to transfer charge q in a capacitor

$$dW = \Delta V\, dq = \frac{q}{C}\, dq$$

$$\Rightarrow W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

The work is the energy that can store in the capacitor.

$$U = \frac{Q^2}{2C} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2$$

For a parallel plate capacitor, $\Delta V = Ed$ & $C = \frac{\epsilon_0 A}{d}$

$$\therefore U = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \frac{\epsilon_0 A}{d} (Ed)^2 = \frac{1}{2} \epsilon_0 A d E^2$$

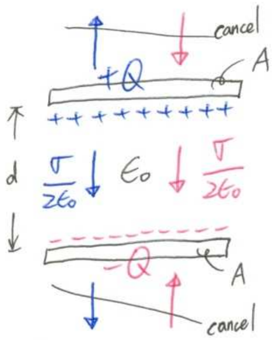
Define energy density $\equiv u$ = energy per unit volume $= \frac{U}{Ad}$

$$\therefore u = \frac{1}{2} \epsilon_0 A d E^2 \frac{1}{Ad}$$

$$= \frac{1}{2} \epsilon_0 E^2 \text{ \#}$$ energy density of the capacitor

---

How to calculate capacitor of a parallel plate?



$$\pm Q = \pm \sigma A$$

$$\vec{E} = \frac{\sigma}{\epsilon_0} (\downarrow) = \frac{Q}{\epsilon_0 A}$$

$$\Delta V = \int_+^- d\vec{l} \cdot \vec{E} = \frac{Q}{\epsilon_0 A} d$$

$$C \equiv \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d} \text{ \#}$$

※ 但本題中，題目未提供 E

所以可選擇以下兩種做法：

Step 1. 推導出 $U = \frac{Q^2}{2C} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2$

Step 2.

[方法①] 若從 $U = \frac{1}{2} Q \Delta V$ 著手（即避開 C）：

$$\Rightarrow u = \frac{U}{Ad} = \frac{Q \Delta V}{2Ad} \quad \#$$

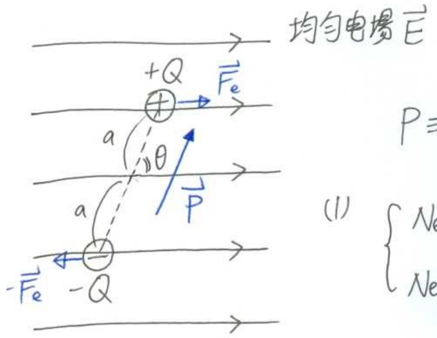
$$= \frac{1}{2} \frac{Q^2}{\epsilon_0 A^2}$$

$\Delta V = \frac{Qd}{\epsilon_0 A}$（利用上頁算出电容的方法推出）

[方法②] 若從 $U = \frac{1}{2} C (\Delta V)^2$ 著手（不避開 C）：

$$\Rightarrow u = \frac{U}{Ad} = \frac{\frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(\Delta V)^2}{Ad} = \frac{1}{2} \epsilon_0 \left(\frac{\Delta V}{d}\right)^2 \quad \#$$

5.



$$P \equiv (2a)\,Q$$

(1) $\begin{cases} \text{Net force is zero.} \\ \text{Net torque is not zero. (The two forces produce a net torque on this dipole.)} \end{cases}$

$$\Rightarrow \boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

$$\Rightarrow \tau = 2Fa\sin\theta$$

$$\underset{F=QE}{=} 2QEa\sin\theta \underset{P=2aQ}{=} PE\sin\theta$$

$$\Rightarrow \vec{\tau} = \vec{P} \times \vec{E} \quad \#$$

(2) The work done to rotate the dipole is stored as potential energy in the system.

$$dW = \tau\, d\theta$$

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau\, d\theta = \int_{\theta_i}^{\theta_f} PE\sin\theta\, d\theta = PE\underbrace{\int_{\theta_i}^{\theta_f} \sin\theta\, d\theta}_{-\cos\theta\big|_{\theta_i}^{\theta_f}} = PE(\cos\theta_i - \cos\theta_f) \quad \#$$

(若令 $U_i = 0$ at $\theta_i = 90^\circ$, 則 $\cos\theta_i = 0$ & $U = U_f$
$= -\vec{P}\cdot\vec{E}$)