The two components are seen to undergo phase jumps of different amounts. Linearly polarized light will in consequence become elliptically polarized on total reflection.

One can also immediately write down an expression for the relative phase difference $\delta = \delta_\perp - \delta_\parallel$:

$$\tan\frac{\delta}{2} = \frac{\tan\frac{\delta_\perp}{2} - \tan\frac{\delta_\parallel}{2}}{1 + \tan\frac{\delta_\perp}{2}\tan\frac{\delta_\parallel}{2}} = \frac{\left(\frac{1}{n^2} - 1\right)\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}}{1 + \frac{\sin^2\theta_i - n^2}{n^2\cos^2\theta_i}},$$

i.e.

$$\tan\frac{\delta}{2} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - n^2}}{\sin^2\theta_i}. \tag{61}$$

This expression vanishes for grazing incidence ($\theta_i = \pi/2$), and for incidence at the critical angle $\bar{\theta}_i$ ($\sin\bar{\theta}_i = n$). Between these two values there lies the maximum value of the relative phase difference; it is determined from the equation

$$\frac{d}{d\theta_i}(\tan\delta/2) = \frac{2n^2 - (1 + n^2)\sin^2\theta_i}{\sin^3\theta_i\sqrt{\sin^2\theta_i - n^2}} = 0.$$

This is satisfied when

$$\sin^2\theta_i = \frac{2n^2}{1 + n^2}. \tag{62}$$

On substituting from (62) into (61), we obtain for the maximum δ_m of the relative phase difference δ, the expression

$$\tan\frac{\delta_m}{2} = \frac{1 - n^2}{2n}. \tag{63}$$

From (61) it is seen that, with n given, there are two values of the angle θ_i of incidence for every value of δ.

The phase change which takes place on total reflection may be used (as shown already by FRESNEL) to produce circularly polarized light from light which is linearly polarized. The amplitude components of the incident light are made equal ($|A_\parallel| = |A_\perp|$), by taking the incident wave to be polarized in a direction which makes an angle of 45° with the normal to the plane of incidence (i.e. $\alpha_i = 45°$). Then, by (58), $|R_\parallel| = |R_\perp|$. Further, n and θ_i are chosen in such a way that the relative phase difference δ is equal to 90°. To attain this value of δ by a single reflection, it would be necessary, according to (63), that

$$\tan\frac{\pi}{4} = 1 < \frac{1 - n^2}{2n},$$

i.e.

$$n = n_{12} < \sqrt{2} - 1 = 0{\cdot}414.$$

This means that the refractive index $n_{21} = 1/n$ of the denser with respect to the less dense medium would have to be at least 2·41. This value is rather large, although there are non-absorbing substances whose refractive index comes close to, and even exceeds, this value. FRESNEL made use of two total reflections on glass. When $n_{21} = 1{\cdot}51$, one obtains, according to (62) and (63), a maximum relative phase difference $\delta_m = 45°\,56'$ when the angle of incidence θ_i equals to $51°\,20'$. It is therefore just possible to attain the value $\delta = 45°$, namely, with either of the following angles of incidence:

$$\theta_i = 48°\,37', \qquad \theta_i = 54°\,37'.$$

A phase difference of 90° may therefore be obtained by means of two successive total reflections at either of these angles. For this purpose a glass block is used, of the form shown in Fig. 1.16, known as *Fresnel's rhomb*.

FRESNEL's rhomb may, of course, be also used to produce elliptically polarized light; the azimuth of the incident (linearly polarized) light must then be taken

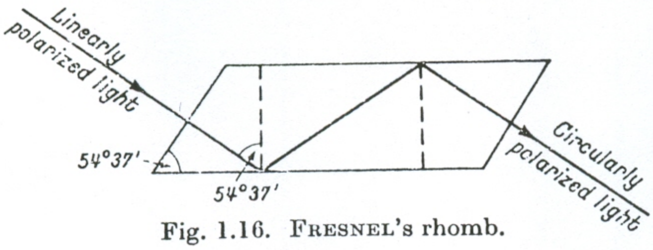

Fig. 1.16. FRESNEL's rhomb.

different from 45°. One may also invert the procedure and produce, by means of FRESNEL's rhomb, linearly polarized light from elliptically polarized light.

Measurement of the critical angle $\bar{\theta}_i$ gives a convenient and accurate way of determining the index of refraction $n = \sin \bar{\theta}_i$. Instruments used for this purpose are called *refractometers*.

1.6 WAVE PROPAGATION IN A STRATIFIED MEDIUM. THEORY OF DIELECTRIC FILMS

A medium whose properties are constant throughout each plane perpendicular to a fixed direction is called a *stratified medium*. If the z-axis of a Cartesian reference system is taken along this special direction, then

$$\varepsilon = \varepsilon(z), \qquad \mu = \mu(z). \tag{1}$$

We shall consider the propagation of a plane, time-harmonic electromagnetic wave through such a medium; this is a natural generalization of the simple case treated in the previous section.

The theory of stratified media is of considerable importance in optics, in connection with *multilayers*, i.e. a succession of thin plane-parallel films. Such films may be produced with the help of high-vacuum evaporation techniques, and their thickness may be controlled with very high accuracy. They have many useful applications. For example, as will be demonstrated later, they may be employed as *antireflection films*, i.e. as coatings which reduce the reflectivity of a given surface. On the other hand thin films will, under appropriate conditions, *enhance* the reflectivity so that when deposited on a glass surface they may be used as beam-splitters, i.e. arrangements employed in interferometry for the division of an incident beam into two parts. Under appropriate conditions a multilayer may also be employed as a filter which transmits (or reflects) only selected regions of the spectrum. Multilayers may also be used as polarizers.

The subject of dielectric and metallic films has been very extensively treated in the scientific literature and many schemes for the computation of the optical effects of multilayers have been proposed. We shall give an outline of the general theory as developed in elegant and important investigations by F. ABELÈS,* and consider in

* F. ABELÈS, *Ann. de Physique*, **5** (1950), 596–640 and 706–782. For a detailed treatment of the subject of thin films see a more specialized treatise e.g. H. MAYER, *Physik dünner Schichten* (Stuttgart, Wissenschaftliche Verlagsgesellschaft, 1950); S. METHFESSEL, *Dünne Schichten* (Halle (Saale), VEB Wilhelm Knapp Verlag, 1953); or O. S. HEAVENS, *Optical Properties of Thin Solid Films* (London, Butterworths Scientific Publications, 1955).

detail some special cases of particular interest. For the treatment of problems involving only a small number of films it is naturally not necessary to use the general theory, and accordingly we shall later (§ 7.6) describe an alternative and older method based on the concept of multiple reflections.

Only dielectric stratified media will be treated in this section. The extension of the analysis to conducting media will be described in § 13.4.

1.6.1 The basic differential equations

Consider a plane, time-harmonic electromagnetic wave propagated through a stratified medium. In the special case when the wave is linearly polarized with its electric vector perpendicular to the plane of incidence we shall speak of a *transverse electric wave* (denoted by TE); when it is linearly polarized with its magnetic vector perpendicular to the plane of incidence we shall speak of a *transverse magnetic wave* (denoted by TM).* Any arbitrarily polarized plane wave may be resolved into two waves, one of which is a TE wave and the other a TM wave. Since according to § 1.5 the boundary conditions at a discontinuity surface for the perpendicular and parallel components are independent of each other, these two waves will also be mutually independent. Moreover, MAXWELL's equations remain unchanged when $\mathbf{E}$ and $\mathbf{H}$ and simultaneously ε and $-\mu$ are interchanged. Thus any theorem relating to TM waves may immediately be deduced from the corresponding result for TE waves by making this change. It will, therefore, be sufficient to study in detail the TE waves only.

We take the plane of incidence to be the yz-plane,† z being the direction of stratification. For a TE wave, $E_y = E_z = 0$ and MAXWELL's equations reduce to the following six scalar equations [time dependence $\exp(-i\omega t)$ being assumed]:

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} + \frac{i\varepsilon\omega}{c} E_x = 0, \qquad (1a) \qquad\qquad \frac{i\omega\mu}{c} H_x = 0, \qquad (2a)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = 0, \qquad (1b) \qquad\qquad \frac{\partial E_x}{\partial z} - \frac{i\omega\mu}{c} H_y = 0, \qquad (2b)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0, \qquad (1c) \qquad\qquad \frac{\partial E_x}{\partial y} + \frac{i\omega\mu}{c} H_z = 0. \qquad (2c)$$

These equations show that H_y, H_z and E_x are functions of y and z only. Eliminating H_y and H_z between (1a), (2b) and (2c) [or by taking the x component of the wave equation § 1.2 (5) for $\mathbf{E}$] it follows that

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + n^2 k_0^2 E_x = \frac{d(\log \mu)}{dz} \frac{\partial E_x}{\partial z}, \qquad (3)$$

where

$$n^2 = \varepsilon\mu, \qquad k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}. \qquad (4)$$

To solve (3) we take, as a trial solution, a product of two functions, one involving y only and the other involving z only:

$$E_x(y, z) = Y(y)U(z). \qquad (5)$$

* The terms "E-polarized" and "H-polarized" are also used (cf. § 11.4.1). It should be mentioned that the terms "transverse electric wave" and "transverse magnetic wave" have different meanings in the theory of wave guides.

† Not the xz-plane as in the previous section.

Eq. (3) then becomes

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{U}\frac{d^2U}{dz^2} - n^2k_0{}^2 + \frac{d(\log\mu)}{dz}\frac{1}{U}\frac{dU}{dz}. \tag{6}$$

Now the term on the left is a function of y only whilst the terms on the right depend only on z. Hence (6) can only hold if each side is equal to a constant ($-K^2$ say):

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -K^2, \tag{7}$$

$$\frac{d^2U}{dz^2} - \frac{d(\log\mu)}{dz}\frac{dU}{dz} + n^2k_0{}^2U = K^2U. \tag{8}$$

It will be convenient to set

$$K^2 = k_0{}^2\alpha^2. \tag{9}$$

Then (7) gives

$$Y = \text{const. } e^{ik_0\alpha y},$$

and consequently E_x is of the form

$$E_x = U(z)e^{i(k_0\alpha y - \omega t)}, \tag{10}$$

where $U(z)$ is a (possibly complex) function of z. From (2b) and (2c) we see that H_y and H_z are given by expressions of the same form:

$$H_y = V(z)e^{i(k_0\alpha y - \omega t)}, \tag{11}$$

$$H_z = W(z)e^{i(k_0\alpha y - \omega t)}. \tag{12}$$

On account of (1a), (2b) and (2c), the amplitude functions U, V and W are related by the following equations:

$$V' = ik_0[\alpha W + \varepsilon U], \tag{13a}$$

$$U' = ik_0\mu V, \tag{13b}$$

$$\alpha U + \mu W = 0, \tag{13c}$$

the prime denoting differentiation with respect to z. Substituting for W from (13c) into (13a) we have, together with (13b), a pair of simultaneous first-order differential equations* for U and V:

$$\left.\begin{aligned} U' &= ik_0\mu V,\\ V' &= ik_0\left(\varepsilon - \frac{\alpha^2}{\mu}\right)U.\end{aligned}\right\} \tag{14}$$

* Eqs. (14) are of the same form as the equations of an electric transmission line, i.e.

$$\frac{dV}{dz} = -ZI, \qquad \frac{dI}{dz} = -YV,$$

where V is the voltage across the line, I is the current in the line, Z is the series impedance, and Y the shunt admittance. The theory of stratified media may therefore be developed in a strict analogy with the theory of electric transmission lines, as has been done by several authors, for example, R. B. Muchmore, *J. Opt. Soc. Amer.*, **38** (1948), 20; K. Schuster, *Ann. d. Physik* (6), **4** (1949), 352; R. Kronig, R. S. Blaisse and J. J. v.v. Sande, *J. Appl. Sci. Res.*, **B,1** (1947), 63.

Elimination between these equations finally gives the following second-order linear differential equations for U and V:

$$\frac{d^2U}{dz^2} - \frac{d\,(\log\mu)}{dz}\frac{dU}{dz} + k_0^2(n^2 - \alpha^2)U = 0, \tag{15}$$

$$\frac{d^2V}{dz^2} - \frac{d\left[\log\left(\varepsilon - \dfrac{\alpha^2}{\mu}\right)\right]}{dz}\frac{dV}{dz} + k_0^2(n^2 - \alpha^2)V = 0. \tag{16}$$

According to the substitution rule which is a consequence of the symmetry of MAXWELL's equations, it immediately follows that *for the TM wave* ($H_y = H_z = 0$), the non-vanishing components of the field vectors are of the form:

$$H_x = U(z)e^{i(k_0\alpha y - \omega t)}, \tag{17}$$

$$E_y = -V(z)e^{i(k_0\alpha y - \omega t)}, \tag{18}$$

$$E_z = -W(z)e^{i(k_0\alpha y - \omega t)} \tag{19}$$

where

$$\left.\begin{aligned} U' &= ik_0\varepsilon V, \\ V' &= ik_0\left(\mu - \frac{\alpha^2}{\varepsilon}\right)U, \end{aligned}\right\} \tag{20}$$

and W is related to U by means of the equation

$$\alpha U + \varepsilon W = 0. \tag{21}$$

U and V now satisfy the following second-order linear differential equations:

$$\frac{d^2U}{dz^2} - \frac{d\{\log\varepsilon\}}{dz}\frac{dU}{dz} + k_0^2(n^2 - \alpha^2)U = 0, \tag{22}$$

$$\frac{d^2V}{dz^2} - \frac{d\left\{\log\left(\mu - \dfrac{\alpha^2}{\varepsilon}\right)\right\}}{dz}\frac{dV}{dz} + k_0^2(n^2 - \alpha^2)V = 0. \tag{23}$$

U, V and W are in general complex functions of z. The surfaces of constant amplitude of E_x are given by

$$|U(z)| = \text{constant},$$

whilst the surfaces of constant phase have the equation

$$\phi(z) + k_0\alpha y = \text{constant},$$

where $\phi(z)$ is the phase of U. The two sets of surfaces do not in general coincide so that E_x (and similarly H_y and H_z) is an inhomogeneous wave. For a small displacement (dy, dz) along a co-phasal surface, $\phi'(z)dz + k_0\alpha dy = 0$; hence if θ denotes the angle which the normal to the co-phasal surface makes with OZ, then

$$\tan\theta = -\frac{dz}{dy} = \frac{k_0\alpha}{\phi'(z)}.$$

In the special case when the wave is an homogeneous plane wave,

$$\phi(z) = k_0 nz \cos \theta, \qquad \alpha = n \sin \theta. \tag{24}$$

Hence the relation

$$\alpha = \text{constant}$$

imposed by (9) may be regarded as a generalization of *Snell's law of refraction* to stratified media.

1.6.2 The characteristic matrix of a stratified medium

The solutions, subject to appropriate boundary conditions, of the differential equations which we have just derived, and various theorems relating to stratified media, can most conveniently be expressed in terms of matrices. We shall therefore give a brief account of the main definitions relating to matrices before discussing the consequences of our equations.

I. By a matrix one understands a system of real or complex numbers, arranged in a rectangular or a square array:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

a_{ij} denoting the element in the ith row and the jth column. The matrix is denoted symbolically by $\mathbf{A}$ or $[a_{ij}]$, and is said to be an m by n matrix (or $m \times n$ matrix), since it contains m rows and n columns. In the special case when $m = n$, $\mathbf{A}$ is said to be a *square matrix* of order m. If $\mathbf{A}$ is a square matrix, the determinant whose elements are the same, and are in the same positions as the elements of $\mathbf{A}$, is said to be the *determinant of the matrix* $\mathbf{A}$; it is denoted by $|\mathbf{A}|$ or $|a_{ij}|$. If $|\mathbf{A}| = 1$, $\mathbf{A}$ is said to be *unimodular*.

By definition two matrices are *equal* only if they have the same number of rows (m) and the same number of columns (n), and if their corresponding elements are equal. If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are two matrices with the same number of rows and the same number of columns, then their *sum* $\mathbf{A} + \mathbf{B}$ is defined as the matrix $\mathbf{C}$ whose elements are $c_{ij} = a_{ij} + b_{ij}$. Similarly their *difference* $\mathbf{A} - \mathbf{B}$ is defined as the matrix $\mathbf{D}$ with elements $d_{ij} = a_{ij} - b_{ij}$.

A matrix having every element zero is called a *null matrix*. The square matrix with elements $a_{ij} = 0$ when $i \neq j$ and $a_{ii} = 1$ for every value of i is called *unit matrix* and will be denoted by $\mathbf{I}$.

The product of a matrix $\mathbf{A}$ and a number λ (real or complex) is defined as the matrix $\mathbf{B}$ with elements $b_{ij} = \lambda a_{ij}$.

The product $\mathbf{AB}$ of two matrices is defined only when the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$. If $\mathbf{A}$ is a $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix the product is then by definition the $m \times n$ matrix with elements

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

The process of multiplication of two matrices is thus analogous to the row-by-column rule for multiplication of determinants of equal orders. In general $\mathbf{AB} \neq \mathbf{BA}$. For example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

whilst

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In the special case when $\mathbf{AB} = \mathbf{BA}$, the matrices $\mathbf{A}$ and $\mathbf{B}$ are said to *commute*.

The above definitions and properties of matrices are the only ones necessary for our purposes, and we can, therefore, now return to our discussion of propagation of electromagnetic waves through a stratified medium.

II. Since the functions $U(z)$ and $V(z)$ of § 1.6.1 each satisfy a second-order linear differential equation [(15) and (16)], it follows that U and V may each be expressed as a linear combination of two particular solutions, say U_1, U_2 and V_1, V_2. These particular solutions cannot be arbitrary; they must be coupled by the first order differential equations (14):

$$\left.\begin{aligned} U_1' &= ik_0\mu V_1, \\ V_1' &= ik_0\left(\varepsilon - \frac{\alpha^2}{\mu}\right) U_1, \end{aligned}\right\} \qquad \left.\begin{aligned} U_2' &= ik_0\mu V_2, \\ V_2' &= ik_0\left(\varepsilon - \frac{\alpha^2}{\mu}\right) U_2. \end{aligned}\right\} \tag{25}$$

From these relations it follows that

$$V_1U_2' - U_1'V_2 = 0, \qquad U_1V_2' - V_1'U_2 = 0,$$

so that

$$\frac{d}{dz}(U_1V_2 - U_2V_1) = 0.$$

This relation implies that *the determinant*

$$D = \begin{vmatrix} U_1 & V_1 \\ U_2 & V_2 \end{vmatrix} \tag{26}$$

associated with any two arbitrary solutions of (14) is a constant, i.e. that D is an invariant of our system of equations.*

For our purposes the most convenient choice of the particular solutions is

$$\left.\begin{aligned} U_1 &= f(z), & U_2 &= F(z), \\ V_1 &= g(z), & V_2 &= G(z), \end{aligned}\right\} \tag{27}$$

such that

$$f(0) = G(0) = 0 \quad \text{and} \quad F(0) = g(0) = 1. \tag{28}$$

---

* This also follows from a well-known property of a Wronskian of second-order linear differential equations. Moreover it may also be shown that, if U_1 is known, U_2 may be obtained by integration from the relation

$$U_2 = ikDU_1\int \frac{\mu}{U_1{}^2}\,dz.$$

Cf. F. Abelès, *Ann. de Physique*, **5** (1950), 603.

Then the solutions with

$$U(0) = U_0, \qquad V(0) = V_0, \tag{29}$$

may be expressed in the form

$$\left.\begin{aligned} U &= FU_0 + fV_0, \\ V &= GU_0 + gV_0, \end{aligned}\right\}$$

or, in matrix notation,

$$\mathbf{Q} = \mathbf{N}\mathbf{Q}_0, \tag{30}$$

where

$$\mathbf{Q} = \begin{bmatrix} U(z) \\ V(z) \end{bmatrix}, \qquad \mathbf{Q}_0 = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} F(z) & f(z) \\ G(z) & g(z) \end{bmatrix}. \tag{31}$$

On account of the relation D = constant, the determinant of the square matrix $\mathbf{N}$ is a constant. The value of this constant may immediately be found by taking $z = 0$, giving

$$|\mathbf{N}| = Fg - fG = 1.$$

It is usually more convenient to express U_0 and V_0 as functions of $U(z)$ and $V(z)$. Solving for U_0 and V_0, we obtain

$$\mathbf{Q}_0 = \mathbf{M}\mathbf{Q}, \tag{32}$$

where

$$\mathbf{M} = \begin{bmatrix} g(z) & -f(z) \\ -G(z) & F(z) \end{bmatrix}. \tag{33}$$

This matrix is also unimodular,

$$|\mathbf{M}| = 1. \tag{34}$$

The significance of $\mathbf{M}$ is clear: it relates the x and y components of the electric (or magnetic) vectors in the plane $z = 0$, to the components in an arbitrary plane z = constant. Now we saw that the knowledge of U and V is sufficient for the complete specification of the field. Hence *for the purposes of determining the propagation of a plane monochromatic wave through a stratified medium, the medium only need be specified by an appropriate two by two unimodular matrix* $\mathbf{M}$. For this reason we shall call $\mathbf{M}$ the *characteristic matrix* of the stratified medium. The constancy of the determinant $|\mathbf{M}|$ may be shown to imply the conservation of energy.*

We shall now consider the form of the characteristic matrix for cases of particular interest.

(a) *A homogeneous dielectric film*

In this case ε, μ and $n = \sqrt{\varepsilon\mu}$ are constants. If θ denotes the angle which the normal to the wave makes with the z-axis, we have by (24),

$$\alpha = n \sin \theta,$$

For a TE wave, we have according to (15) and (16),

$$\left.\begin{aligned} \frac{d^2U}{dz^2} + (k_0^2 n^2 \cos^2\theta)U &= 0, \\ \frac{d^2V}{dz^2} + (k_0^2 n^2 \cos^2\theta)V &= 0. \end{aligned}\right\} \tag{35}$$

* To show this, one evaluates the reflectivity and transmissivity (51) in terms of the matrix elements. If further one uses the fact that for a non-absorbing medium the characteristic matrix is of the form indicated by (45), it follows that the conservation law $\mathscr{R} + \mathscr{T} = 1$ will be satisfied provided that $|\mathbf{M}| = 1$.

The solutions of these equations, subject to the relations (14), are easily seen to be

$$\left.\begin{aligned} U(z) &= A\cos(k_0nz\cos\theta) + B\sin(k_0nz\cos\theta),\\ V(z) &= \frac{1}{i}\sqrt{\frac{\varepsilon}{\mu}}\cos\theta\{B\cos(k_0nz\cos\theta) - A\sin(k_0nz\cos\theta)\}.\end{aligned}\right\} \tag{36}$$

Hence the particular solutions (27) which satisfy the boundary conditions (28) are

$$\left.\begin{aligned} U_1 = f(z) &= \frac{i}{\cos\theta}\sqrt{\frac{\mu}{\varepsilon}}\sin(k_0nz\cos\theta),\\ V_1 = g(z) &= \cos(k_0nz\cos\theta),\\ U_2 = F(z) &= \cos(k_0nz\cos\theta),\\ V_2 = G(z) &= i\sqrt{\frac{\varepsilon}{\mu}}\cos\theta\sin(k_0nz\cos\theta).\end{aligned}\right\} \tag{37}$$

If we set

$$p = \sqrt{\frac{\varepsilon}{\mu}}\cos\theta, \tag{38}$$

the characteristic matrix is seen to be

$$\mathbf{M}(z) = \begin{bmatrix} \cos(k_0nz\cos\theta) & -\frac{i}{p}\sin(k_0nz\cos\theta)\\ -ip\sin(k_0nz\cos\theta) & \cos(k_0nz\cos\theta)\end{bmatrix}. \tag{39}$$

For a *TM* wave, the same equations hold, with p replaced by

$$q = \sqrt{\frac{\mu}{\varepsilon}}\cos\theta. \tag{40}$$

(b) *A stratified medium as a pile of thin homogeneous films*

Consider two adjacent stratified media, the first one extending from $z = 0$ to $z = z_1$, and the second from $z = z_1$ to $z = z_2$. If $\mathbf{M}_1(z)$ and $\mathbf{M}_2(z)$ are the characteristic matrices of the two media, then

$$\mathbf{Q}_0 = \mathbf{M}_1(z_1)\mathbf{Q}(z_1), \qquad \mathbf{Q}(z_1) = \mathbf{M}_2(z_2 - z_1)\mathbf{Q}(z_2),$$

so that

$$\mathbf{Q}_0 = \mathbf{M}(z_2)\mathbf{Q}(z_2),$$

where

$$\mathbf{M}(z_2) = \mathbf{M}_1(z_1)\mathbf{M}_2(z_2 - z_1).$$

This result may immediately be generalized to the case of a succession of stratified media extending from $0 \leqslant z \leqslant z_1, z_1 \leqslant z \leqslant z_2, \ldots z_{N-1} \leqslant z \leqslant z_N$. If the characteristic matrices are $\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_N$, then

$$\left.\begin{aligned} &\mathbf{Q}_0 = \mathbf{M}(z_N)\mathbf{Q}(z_N),\\ \text{where}\quad &\mathbf{M}(z_N) = \mathbf{M}_1(z_1)\mathbf{M}_2(z_2 - z_1)\ldots\mathbf{M}_N(z_N - z_{N-1}).\end{aligned}\right\} \tag{41}$$

With the help of (41) an approximate expression for the characteristic matrix of any stratified medium may easily be derived*: we regard the medium as consisting of a very large number of thin films of thickness $\delta z_1, \delta z_2, \delta z_3, \ldots \delta z_n$. If the maximum

* For a fuller treatment of stratified media of continuously varying refractive index see R. Jacobsson, *Progress in Optics*, Vol. 5, ed. E. Wolf (Amsterdam, North Holland Publishing Company and New York, J. Wiley and Sons, 1965), p. 247.

thickness is sufficiently small, it is permissible to regard ε, μ and n to be constant throughout each film. From (39) it is seen that the characteristic matrix of the jth film is then approximately given by

$$\mathbf{M}_j = \begin{bmatrix} 1 & -\dfrac{i}{p_j} k_0 n_j \delta z_j \cos\theta_j \\ -ip_j k_0 n_j \delta z_j \cos\theta_j & 1 \end{bmatrix}.$$

Hence the characteristic matrix of the whole medium, considered as a pile of thin films, is approximately equal to (again retaining terms up to the first power in δz only):

$$\mathbf{M} = \prod_{j=1}^{N} \mathbf{M}_j = \begin{bmatrix} 1 & -ik_0 B \\ -ik_0 A & 1 \end{bmatrix}, \tag{42}$$

where

$$A = \sum_{j=1}^{N} p_j n_j \delta z_j \cos\theta_j = \sum_{j=1}^{N} \left(\varepsilon_j - \frac{\alpha^2}{\mu_j}\right) \delta z_j,$$

$$B = \sum_{j=1}^{N} \frac{n_j}{p_j} \delta z_j \cos\theta_j = \sum_{j=1}^{N} \mu_j \delta z_j.$$

Proceeding to the limit as $N \to \infty$ in such a way that $\max |\delta z_j| \to 0$, we obtain

$$\mathbf{M} = \begin{bmatrix} 1 & -ik_0 \mathscr{B} \\ -ik_0 \mathscr{A} & 1 \end{bmatrix}, \tag{43}$$

where

$$\mathscr{A} = \int \left(\varepsilon - \frac{\alpha^2}{\mu}\right) dz, \qquad \mathscr{B} = \int \mu dz, \tag{44}$$

the integration being taken throughout the whole z range. Eq. (43) gives a first approximation to the characteristic matrix of an arbitrary stratified medium. Improved approximations can be obtained by retaining higher-order terms* in the expansions of $\cos(k_0 n \delta z \cos\theta)$ and $\sin(k_0 n \delta z \cos\theta)$ and in the product (42).

Since, for a non-absorbing medium, ε and μ are real, it is also seen that *the characteristic matrix of a non-absorbing stratified medium has the form*

$$\mathbf{M} = \begin{bmatrix} a & ib \\ ic & d \end{bmatrix}, \tag{45}$$

where a, b, c and d are real.

1.6.3 The reflection and transmission coefficients

Consider a plane wave incident upon a stratified medium that extends from $z = 0$ to $z = z_1$ and that is bounded on each side by a homogeneous, semi-infinite medium. We shall derive expressions for the amplitudes and intensities of the reflected and transmitted waves.†

Let A, R and T denote as before the amplitudes (possibly complex) of the electric vectors of the incident, reflected and transmitted waves. Further, let ε_1, μ_1 and ε_l, μ_l be the dielectric constant and the magnetic permeability of the first and the last

* This is discussed fully in the paper by F. Abelès, *Ann. d Physique*, **5** (1950), p. 611.

† We consider the amplitudes of the electric vectors when studying a *TE* wave and those of the magnetic vectors when studying a *TM* wave.

medium, and let θ_1 and θ_l be the angles which the normals to the incident and the transmitted waves make with the z-direction (direction of stratification).

The boundary conditions of § 1.1 demand that the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ shall be continuous across each of the two boundaries of the stratified medium. This gives, if the relation § 1.4 (4)

$$\boldsymbol{H} = \sqrt{\frac{\varepsilon}{\mu}}\, \boldsymbol{s} \wedge \boldsymbol{E}$$

is also used, the following relations for a TE wave:

$$\left.\begin{aligned} U_0 &= A + R, & U(z_1) &= T, \\ V_0 &= p_1(A - R), & V(z_1) &= p_l T, \end{aligned}\right\} \tag{46}$$

where

$$p_1 = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_1, \qquad p_l = \sqrt{\frac{\varepsilon_l}{\mu_l}} \cos\theta_l. \tag{47}$$

The four quantities U_0, V_0, U and V given by (46) are connected by the basic relation (32); hence

$$\left.\begin{aligned} A + R &= (m'_{11} + m'_{12}p_l)T, \\ p_1(A - R) &= (m'_{21} + m'_{22}p_l)T, \end{aligned}\right\} \tag{48}$$

m'_{ij} being the elements of the characteristic matrix of the medium, evaluated for $z = z_1$.

From (48) we obtain the reflection and transmission coefficients of the film:

$$r = \frac{R}{A} = \frac{(m'_{11} + m'_{12}p_l)p_1 - (m'_{21} + m'_{22}p_l)}{(m'_{11} + m'_{12}p_l)p_1 + (m'_{21} + m'_{22}p_l)}, \tag{49}$$

$$t = \frac{T}{A} = \frac{2p_1}{(m'_{11} + m'_{12}p_l)p_1 + (m'_{21} + m'_{22}p_l)}. \tag{50}$$

In terms of r and t, the *reflectivity* and *transmissivity* are

$$\mathscr{R} = |r|^2, \qquad \mathscr{T} = \frac{p_l}{p_1}|t|^2. \tag{51}$$

The phase δ_r of r may be called *the phase change on reflection* and the phase δ_t of t *the phase change on transmission*. The phase change δ_r is referred to the first surface of discontinuity, whilst the phase change δ_t is referred to the plane boundary between the stratified medium and the last semi-infinite medium.

The corresponding formulae for a TM wave are immediately obtained from (49)–(51) on replacing the quantities p_1 and p_l by

$$q_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_1, \qquad q_l = \sqrt{\frac{\mu_l}{\varepsilon_l}} \cos\theta_l. \tag{52}$$

r and t are then the ratios of the amplitudes of the magnetic and not the electric vectors.

1.6.4 A homogeneous dielectric film*

The properties of an homogeneous dielectric film situated between two homogeneous media is of particular interest in optics, and we shall, therefore, study this case more fully. We assume all the media to be non-magnetic ($\mu = 1$).

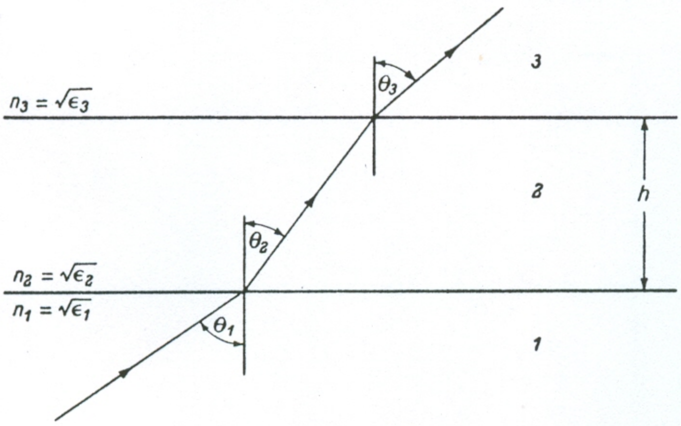

Fig. 1.17. Propagation of an electromagnetic wave through a homogeneous film.

The characteristic matrix of a homogeneous dielectric film is given by (39). Denoting by subscripts 1, 2 and 3 quantities which refer to the three media (see Fig. 1.17), and by h the thickness of the film, we have

$$m'_{11} = m'_{22} = \cos\beta, \qquad m'_{12} = -\frac{i}{p_2}\sin\beta, \qquad m'_{21} = -ip_2\sin\beta, \tag{53}$$

where

$$\beta = \frac{2\pi}{\lambda_0} n_2 h \cos\theta_2$$

and

$$p_j = n_j\cos\theta_j, \qquad (j = 1, 2, 3). \tag{54}$$

The reflection and transmission coefficients r and t may be obtained by substituting these expressions into (49) and (50), with $l = 3$. The resulting formulae may be conveniently expressed in terms of the corresponding coefficients r_{12}, t_{12} and r_{23}, t_{23} associated with the reflection and transmission at the first and the second surface respectively. According to the FRESNEL formulae § 1.5 (20) and (21) we have for a TE wave,

$$r_{12} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{p_1 - p_2}{p_1 + p_2}, \tag{55}$$

$$t_{12} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{2p_1}{p_1 + p_2}, \tag{56}$$

* An alternative derivation of the main formulae relating to the properties of a single dielectric film will be found in § 7.6.1. The formulae may, of course, also be derived directly by applying the boundary conditions of § 1.1.3 at each boundary of the film (cf. M. BORN, *Optik* (Berlin, Springer, 1933), p. 125; or H. MAYER, *Physik dünner Schichten* (Stuttgart, Wissenschaftliche Verlagsgesellschaft 1950), p. 145.

with analogous expressions for r_{23} and t_{23}. In terms of these expressions, the formulae for r and t become*

$$r = \frac{r_{12} + r_{23}e^{2i\beta}}{1 + r_{12}r_{23}e^{2i\beta}}, \tag{57}$$

$$t = \frac{t_{12}t_{23}e^{i\beta}}{1 + r_{12}r_{23}e^{2i\beta}}; \tag{58}$$

the reflectivity and transmissivity are therefore given by

$$\mathscr{R} = |r|^2 = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23}\cos 2\beta}{1 + r_{12}^2r_{23}^2 + 2r_{12}r_{23}\cos 2\beta}; \tag{59}$$

and

$$\mathscr{T} = \frac{p_3}{p_1}|t|^2 = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}\frac{t_{12}^2t_{23}^2}{1 + r_{12}^2r_{23}^2 + 2r_{12}r_{23}\cos 2\beta}. \tag{60}$$

A straightforward calculation gives, as expected,

$$\mathscr{R} + \mathscr{T} = 1.$$

The phase changes can also easily be calculated from (57) and (58), and are found to be given by

$$\tan\delta_r = \tan(\arg r) = \frac{r_{23}(1 - r_{12}^2)\sin 2\beta}{r_{12}(1 + r_{23}^2) + r_{23}(1 + r_{12}^2)\cos 2\beta}, \tag{61}$$

$$\tan\delta_t = \tan(\arg t) = \frac{1 - r_{12}r_{23}}{1 + r_{12}r_{23}}\tan\beta. \tag{62}$$

Let us now briefly consider the implications of these formulae. We first note that (59) and (60) remain unchanged when β is replaced by $\beta + \pi$, i.e. when h is replaced by $h + \Delta h$, where

$$\Delta h = \frac{\lambda_0}{2n_2\cos\theta_2}. \tag{63}$$

Hence *the reflectivity and transmissivity of dielectric films which differ in thickness by an integral multiple of* $\lambda_0/2n_2\cos\theta_2$ *are the same.*

Next we determine the optical thickness for which the reflection coefficient has a maximum or a minimum. If we set

$$H = n_2 h, \tag{64}$$

we find from (59) that

$$\frac{d\mathscr{R}}{dH} = 0 \qquad \text{when} \qquad \sin 2\beta = 0,$$

i.e. when

$$H = \frac{m\lambda_0}{4\cos\theta_2}, \qquad (m = 0, 1, 2, \ldots).$$

We must distinguish two cases:

(1) When *m is odd*, i.e. when H has any of the values

$$H = \frac{\lambda_0}{4\cos\theta_2}, \quad \frac{3\lambda_0}{4\cos\theta_2}, \quad \frac{5\lambda_0}{4\cos\theta_2}, \ldots$$

* These formulae were first derived in a different manner by G. B. AIRY, *Phil. Mag.*, **2** (1833), 20; also *Ann. Phys. und Chem.* (Ed. Poggendorf), **41** (1837), 512.

then $\cos 2\beta = -1$ and (59) reduces to

$$\mathscr{R} = \left(\frac{r_{12} - r_{23}}{1 - r_{12}r_{23}}\right)^2. \tag{65}$$

In particular for *normal incidence*, one has from (55)

$$r_{12} = \frac{n_1 - n_2}{n_1 + n_2}, \qquad r_{23} = \frac{n_2 - n_3}{n_2 + n_3}, \tag{66}$$

and (65) becomes

$$\mathscr{R} = \left(\frac{n_1 n_3 - n_2^2}{n_1 n_3 + n_2^2}\right)^2. \tag{67}$$

(2) When *m is even*, i.e. when the optical thickness has any of the values

$$H = \frac{\lambda_0}{2\cos\theta_2}, \qquad \frac{2\lambda_0}{2\cos\theta_2}, \qquad \frac{3\lambda_0}{2\cos\theta_2}, \ldots$$

then $\cos 2\beta = 1$ and (59) reduces to

$$\mathscr{R} = \left(\frac{r_{12} + r_{23}}{1 + r_{12}r_{23}}\right)^2. \tag{68}$$

In particular, for *normal incidence*, this becomes

$$\mathscr{R} = \left(\frac{n_1 - n_3}{n_1 + n_3}\right)^2, \tag{69}$$

and is seen to be independent of n_2. Now the only difference in the case of oblique incidence is the replacement of n_j by $n_j \cos\theta_j$ $(j = 1, 2, 3)$ in all the formulae; hence *a plate whose optical thickness is* $m\lambda_0/2\cos\theta_2$ $(m = 1, 2, 3 \ldots)$ *has no influence on the intensity of the reflected (or transmitted) radiation.*

Next we must determine the nature of these extreme values. After a straightforward calculation we find that when $H = m\lambda_0/4\cos\theta_2$ $(m = 1, 2, \ldots)$

$$\left.\begin{aligned} &\left(\frac{d^2\mathscr{R}}{dH^2}\right) \gtrless 0 \\ \text{according as}\quad &(-1)^m r_{12}r_{23}[1 + r_{12}^2 r_{23}^2 - r_{12}^2 - r_{23}^2] \lessgtr 0, \end{aligned}\right\} \tag{70}$$

so that with the upper sign there is a minimum and with the lower sign a maximum. In particular, for *normal incidence*, r_{12} and r_{23} are given by (66) and we have

$$\left.\begin{aligned} &\textit{maximum}, \text{ if } \quad (-1)^m(n_1 - n_2)(n_2 - n_3) > 0, \\ &\textit{minimum}, \text{ if } \quad (-1)^m(n_1 - n_2)(n_2 - n_3) < 0. \end{aligned}\right\} \tag{71}$$

Usually the first medium is air $(n_1 \sim 1)$ and we see that *with a film whose optical thickness has any of the values* $\lambda_0/4, 3\lambda_0/4, 5\lambda_0/4, \ldots$ *the reflectivity is then a maximum or a minimum according to whether the refractive index of the film is greater or smaller than the refractive index of the last medium; for a film whose optical thickness has any of the values* $\lambda_0/2, 2\lambda_0/2, 3\lambda_0/2, \ldots$ *the opposite is the case.*

These results, which are illustrated in Fig. 1.18, are found to be in good agreement with experiment.*

It is evident from the preceding analysis that a plate whose optical thickness is a quarter of the wavelength and whose refractive index is low enough may be used as an *antireflection film*, i.e. a film by means of which the reflectivity of a surface is reduced. (The surface is then said to be "bloomed".) The two substances most commonly used for this purpose are cryolite ($n \sim 1{\cdot}35$) and magnesium fluoride (MgF_2, $n \sim 1{\cdot}38$)†:

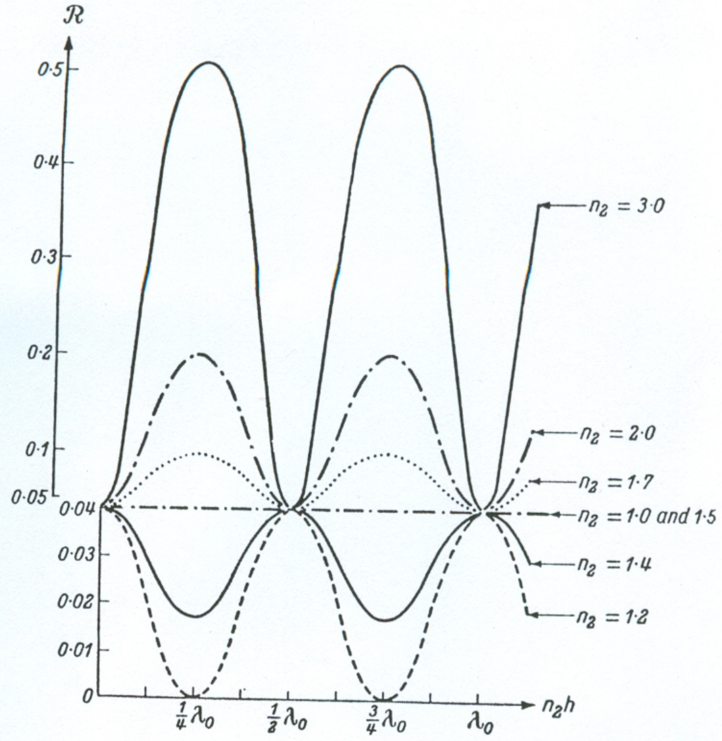

Fig. 1.18. The reflectivity of a dielectric film of refractive index n_2 as a function of its optical thickness. ($\theta_1 = 0$, $n_1 = 1$, $n_3 = 1{\cdot}5$).
(After R. Messner, *Zeiss Nachr.*, **4** (H9) (1943), 253.)

According to (67), the reflectivity at normal incidence would be strictly zero if

$$n_2 = \sqrt{n_1 n_3}. \tag{72}$$

With $n_1 = 1$, $n_3 = 1{\cdot}5$ this demands $n_2 \sim 1{\cdot}22$, a condition which cannot be satisfied in practice. A fuller analysis of (59) shows, however, that with oblique incidence it is possible to have zero reflectivity for a TM wave (electric vector parallel to the plane of incidence) but not for a TE wave (electric vector perpendicular to the plane of incidence), i.e. under favourable conditions one can have simultaneously $\mathscr{R}_\parallel = 0, \mathscr{R}_\perp \neq 0$. Hence a thin film of a suitable dielectric material may also be used as a *polarizer*, working by reflection. Such a polarizer may be regarded as a generalization of the

* See, for example, K. Hammer, *Z. tech. Phys.*, **24** (1943), 169.

† The design and performance of multilayer antireflection films is discussed by A. Musset and A. Thelen in *Progress in Optics*, vol. 8, ed. E. Wolf (Amsterdam, North-Holland Publishing Company and New York, American Elsevier Publishing Company, 1970), p. 201.

simple arrangement discussed earlier in connection with BREWSTER's angle. To obtain a large value for $\mathscr{R}_\perp$ (with $\mathscr{R}_\parallel = 0$), the refractive index n_2 of the film must be as large as possible.* For example, with $n_1 = 1$, $n_2 = 2{\cdot}5$, $n_3 = 1{\cdot}53$ one obtains $\mathscr{R}_\parallel = 0$, $\mathscr{R}_\perp = 0{\cdot}79$ when $\theta_1 = 74°\ 30'$.

If a glass surface is coated with a material of sufficiently high refractive index, the reflectivity of the surface will, according to the preceding analysis, be greatly enhanced (see Figs. 1.18 and 1.19). The surface will then act as a good beam splitter. Coatings of titanium dioxide (TiO_2, $n \sim 2{\cdot}45$) or zinc sulphide (ZnS, $n \sim 2{\cdot}3$) are very suitable for this purpose, giving a maximum reflectivity of about 0·3. There are other substances which have high refractive indices, but they absorb some of the incident light. For example, with a coating of stibnite (Sb_2S_3, $n \sim 2{\cdot}8$) one can attain the values $\mathscr{R} = \mathscr{T} = 0{\cdot}46$, but 8 per cent of the incident light is then absorbed by the film.

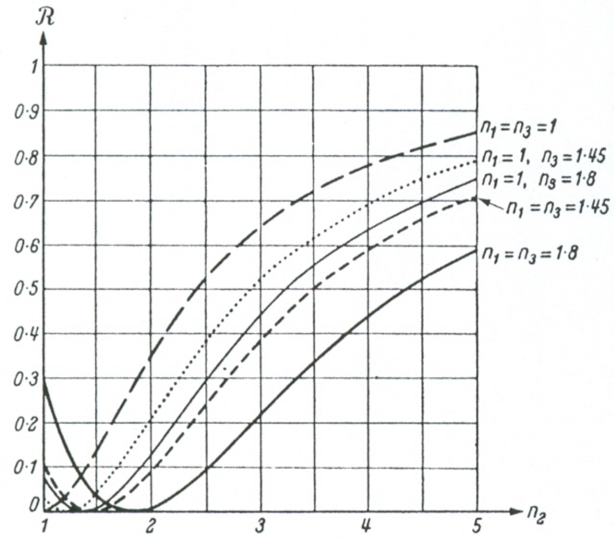

Fig. 1.19. The reflectivity at normal incidence of a quarter-wave film ($n_2 h = \lambda_0/4$) as function of the refractive index n_2 of the film. (After K. HAMMER, *Z. tech. Phys.*, **24** (1943), 169.)

It is also of interest to examine the case when total reflection takes place on the first boundary. In this case

$$n_1 \sin\theta_1 > n_2, \qquad n_1 \sin\theta_1 < n_3,$$

and (cf. § 1.5, eq. 54),

$$n_2 \cos\theta_2 = i\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}. \tag{73}$$

The coefficients for reflection at the two boundaries now are (cf. § 1.5 (21))

$$\left.\begin{aligned} r_{12} &= \frac{n_1\cos\theta_1 - i\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1 + i\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}, \\ r_{23} &= \frac{i\sqrt{n_1^2\sin^2\theta_1 - n_2^2} - n_3\cos\theta_3}{i\sqrt{n_1^2\sin^2\theta_1 - n_2^2} + n_3\cos\theta_3}. \end{aligned}\right\} \tag{74}$$

If we set

$$k_0 n_2 h \cos\theta_2 = ib, \tag{75}$$

* Cf. H. SCHRÖDER, *Optik*, **3** (1948), 499.

where, according to (73),

$$b = \frac{2\pi}{\lambda_0} h\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}, \tag{76}$$

we obtain for the reflection coefficients the following expression, in place of (57):

$$r = \frac{r_{12} + r_{23}e^{-2b}}{1 + r_{12}r_{23}e^{-2b}}. \tag{77}$$

Since $|r_{12}| = |r_{23}| = 1$, r_{12} and r_{23} are of the form

$$r_{12} = e^{i\phi_{12}}, \qquad r_{23} = e^{i\phi_{23}}, \tag{78}$$

where the ϕ's are real; hence the reflectivity now is

$$\mathscr{R} = |r|^2 = \frac{e^{2b} + e^{-2b} + 2\cos(\phi_{12} - \phi_{23})}{e^{2b} + e^{-2b} + 2\cos(\phi_{12} + \phi_{23})}. \tag{79}$$

In contrast with the previous case, $\mathscr{R}$ is now no longer a periodic function of the thickness of the film. (76) shows that if the dependence of the refractive index on the wavelength is neglected, b is inversely proportional to wavelength. Since for sufficiently large values of b, $\mathscr{R}$ will be practically unity, the shorter wavelengths will not be transmitted; the film then acts as a *low-pass filter*, i.e. one which transmits the long wavelengths only.

We have seen that by the use of dielectric films of suitable material, many useful effects can be attained. It will be apparent that, with a number of such films arranged in succession, the desired features may be still further enhanced. The characteristic matrix of such a *multilayer* may be obtained with the help of the theorem expressed by (41).* We shall discuss in detail only the case when the multilayer is periodic.

1.6.5 Periodically stratified media

A stratified periodic medium with period h is characterized by a dielectric constant ε and a magnetic permeability μ which are functions of z only and are such that

$$\varepsilon(z + jh) = \varepsilon(z), \qquad \mu(z + jh) = \mu(z),$$

being any integer in some fixed range $1 \leqslant j \leqslant N$.

Let $\mathbf{M}(h)$ be the characteristic matrix corresponding to one period and write†

$$\mathbf{M}(h) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}. \tag{80}$$

According to (41) we then have, on account of the periodicity,

$$\mathbf{M}(Nh) = \underbrace{\mathbf{M}(h)\,.\,\mathbf{M}(h)\,.\,.\,.\,\mathbf{M}(h)}_{N \text{ times}} = (\mathbf{M}(h))^N. \tag{81}$$

---

* Formulae relating to multilayers have been given by many writers, e.g. R. L. MOONEY, *J. Opt. Soc. Amer.*, **36** (1946), 256; W. WEINSTEIN, *ibid*, **37** (1947), 576.

† We now omit the prime on the matrix elements.

To evaluate the elements of the matrix $\mathbf{M}(Nh)$ we use a result from the theory of matrices, according to which the Nth power of a unimodular matrix $\mathbf{M}(h)$ is*

$$[\mathbf{M}(h)]^N = \begin{bmatrix} m_{11}\mathcal{U}_{N-1}(a) - \mathcal{U}_{N-2}(a) & m_{12}\mathcal{U}_{N-1}(a) \\ m_{21}\mathcal{U}_{N-1}(a) & m_{22}\mathcal{U}_{N-1}(a) - \mathcal{U}_{N-2}(a) \end{bmatrix}, \tag{82}$$

where

$$a = \tfrac{1}{2}(m_{11} + m_{22}), \tag{83}$$

and $\mathcal{U}_N$ are the *Chebyshev Polynomials* of the second kind†:

$$\mathcal{U}_N(x) = \frac{\sin[(N+1)\cos^{-1} x]}{\sqrt{1-x^2}}. \tag{84}$$

A multilayer usually consists of a succession of homogeneous layers of alternately low and high refractive indices n_2 and n_3 and of thickness h_2 and h_3, placed between two homogeneous media of refractive indices n_1 and n_l. (See Fig. 1.20.) We again assume the media to be non-magnetic ($\mu = 1$) and set

$$\left.\begin{aligned} \beta_2 &= \frac{2\pi}{\lambda_0} n_2 h_2 \cos\theta_2, & \beta_3 &= \frac{2\pi}{\lambda_0} n_3 h_3 \cos\theta_3, \\ p_2 &= n_2 \cos\theta_2, & p_3 &= n_3\cos\theta_3. \\ & & h = h_2 + h_3. & \end{aligned}\right\} \tag{85}$$

The characteristic matrix $\mathbf{M}_2(h)$ of one period then is, according to (39) and (41),

$$\mathbf{M}_2(h) = \begin{bmatrix} \cos\beta_2 & -\frac{i}{p_2}\sin\beta_2 \\ -ip_2\sin\beta_2 & \cos\beta_2 \end{bmatrix} \begin{bmatrix} \cos\beta_3 & -\frac{i}{p_3}\sin\beta_3 \\ -ip_3\sin\beta_3 & \cos\beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta_2\cos\beta_3 - \frac{p_3}{p_2}\sin\beta_2\sin\beta_3 & -\frac{i}{p_3}\cos\beta_2\sin\beta_3 - \frac{i}{p_2}\sin\beta_2\cos\beta_3 \\ -ip_2\sin\beta_2\cos\beta_3 - ip_3\cos\beta_2\sin\beta_3 & \cos\beta_2\cos\beta_3 - \frac{p_2}{p_3}\sin\beta_2\sin\beta_3 \end{bmatrix}. \tag{86}$$

---

* The correctness of this result may be verified by induction, using the recurrence relation

$$\mathcal{U}_j(x) = 2x\mathcal{U}_{j-1}(x) - \mathcal{U}_{j-2}(x),$$

which follows as an identity from the definition of the Chebyshev Polynomials.

A direct proof based on the theory of matrices was given by F. ABELÈS, *Ann. de Physique*, **5** (1950), 777.

† These polynomials satisfy the following orthogonality and normalizing conditions:

$$\int_{-1}^{+1} \mathcal{U}_m(x)\mathcal{U}_n(x)\sqrt{1-x^2}\,dx = 0 \text{ when } n \neq m$$
$$= \frac{\pi}{2} \text{ when } n = m.$$

For convenience we note the explicit expressions of the first six polynomials:

$$\begin{aligned} \mathcal{U}_0(x) &= 1, & \mathcal{U}_3(x) &= 8x^3 - 4x, \\ \mathcal{U}_1(x) &= 2x, & \mathcal{U}_4(x) &= 16x^4 - 12x^2 + 1, \\ \mathcal{U}_2(x) &= 4x^2 - 1, & \mathcal{U}_5(x) &= 32x^5 - 32x^3 + 6x. \end{aligned}$$

Tables of Chebyshev polynomials have been published by the National Bureau of Standards Washington (Applied Mathematics Series 9 (1952)), where the main properties of the polynomials are also summarized. See also *Higher Transcendental Functions* (Bateman Manuscript Project, New York, McGraw-Hill, Vol. **2** (1953), p. 183).

Hence according to (81), the characteristic matrix $\mathbf{M}_{2N}(Nh)$ of the multilayer (with $2N$ films in all) is given by the following formula due to ABELÈS:

$$\mathbf{M}_{2N}(Nh) = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}, \tag{87}$$

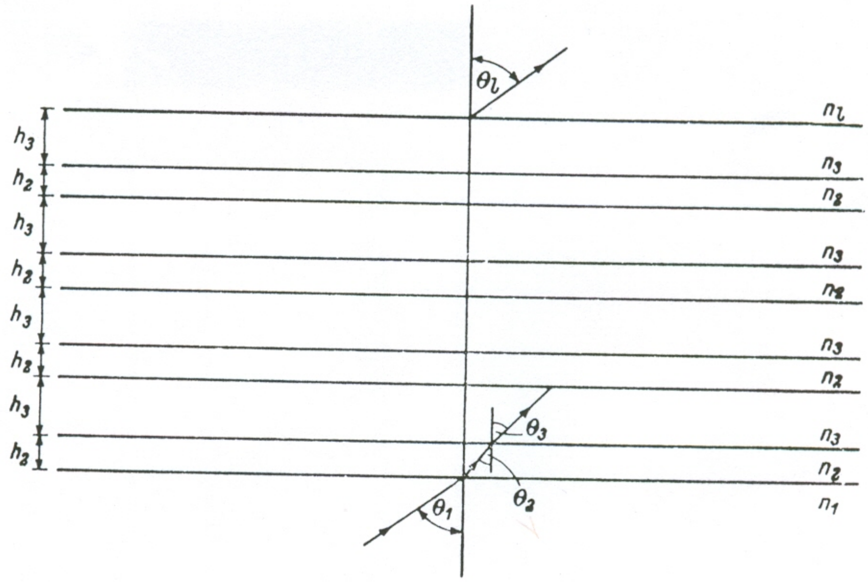

Fig. 1.20. A periodic multilayer.

where

$$\left.\begin{aligned} \mathcal{M}_{11} &= \left[\cos\beta_2\cos\beta_3 - \frac{p_3}{p_2}\sin\beta_2\sin\beta_3\right]\mathcal{U}_{N-1}(a) - \mathcal{U}_{N-2}(a), \\ \mathcal{M}_{12} &= -i\left[\frac{1}{p_3}\cos\beta_2\sin\beta_3 + \frac{1}{p_2}\sin\beta_2\cos\beta_3\right]\mathcal{U}_{N-1}(a), \\ \mathcal{M}_{21} &= -i[p_2\sin\beta_2\cos\beta_3 + p_3\cos\beta_2\sin\beta_3]\mathcal{U}_{N-1}(a), \\ \mathcal{M}_{22} &= \left[\cos\beta_2\cos\beta_3 - \frac{p_2}{p_3}\sin\beta_2\sin\beta_3\right]\mathcal{U}_{N-1}(a) - \mathcal{U}_{N-2}(a), \end{aligned}\right\} \tag{88}$$

and

$$a = \cos\beta_2\cos\beta_3 - \tfrac{1}{2}\left(\frac{p_2}{p_3} + \frac{p_3}{p_2}\right)\sin\beta_2\sin\beta_3. \tag{89}$$

The reflection and transmission coefficients of the multilayer are immediately obtained by substituting these expressions into (49) and (50).

Of particular interest is the case when the two basic layers are of the same optical thickness (usually $\lambda_0/4$), i.e. when

$$n_2h_2 = n_3h_3, \tag{90}$$

and the incidence is normal ($\theta_1 = 0$). Then

$$\beta_2 = \beta_3 = \frac{2\pi}{\lambda_0}n_2h_2 = \frac{2\pi}{\lambda_0}n_3h_3, \tag{91}$$

and if we denote this common value by β, the argument of the CHEBYSHEV polynomials reduces to

$$a = \cos^2 \beta - \tfrac{1}{2}\left(\frac{n_2}{n_3} + \frac{n_3}{n_2}\right)\sin^2 \beta. \tag{92}$$

It is seen that a cannot exceed unity, but that for some values of β it may become smaller than -1. Then $\cos^{-1} a$ will be imaginary and consequently, since for any χ,

$$\sin i\chi = i \sinh \chi = i\,\frac{e^{\chi} - e^{-\chi}}{2}$$

$\mathscr{U}_N$ will have exponential behaviour. It follows that the reflectivity of such a multilayer will increase rapidly with the number of the periods.

With *quarter-wave films* ($n_2 h_2 = n_3 h_3 = \lambda_0/4$) at normal incidence (again assuming non-magnetic media),

$$\beta = \pi/2, \qquad p_2 = n_2, \qquad p_3 = n_3, \tag{93}$$

and (86) reduces to

$$\mathbf{M}_2(h) = \begin{bmatrix} -\dfrac{n_3}{n_2} & 0 \\ 0 & -\dfrac{n_2}{n_3} \end{bmatrix}. \tag{94}$$

The characteristic matrix (87) of the multilayer whose basic period is such a double layer is, as can be directly verified by multiplying (94) N times by itself,

$$\mathbf{M}_{2N}(Nh) = \begin{bmatrix} \left(-\dfrac{n_3}{n_2}\right)^N & 0 \\ 0 & \left(-\dfrac{n_2}{n_3}\right)^N \end{bmatrix}. \tag{95}$$

According to (49) and (51) the reflectivity is

$$\mathscr{R}_{2N} = \left(\frac{1 - \dfrac{n_l}{n_1}\left(\dfrac{n_2}{n_3}\right)^{2N}}{1 + \dfrac{n_l}{n_1}\left(\dfrac{n_2}{n_3}\right)^{2N}} \right)^2. \tag{96}$$

This shows that for a fixed number N of the double layers,* $\mathscr{R}_{2N}$ increases when the ratio n_2/n_3 is increased, and that if this ratio is fixed $\mathscr{R}_{2N}$ increases with N.

Sometimes, for example, for plate coatings of the FABRY–PEROT Interferometer (cf. § 7.6) the layers are arranged in succession characterized by the sequence n_2, n_3, n_2, n_3, . . . n_2, n_3, n_2 of refractive indices. The characteristic matrix of this multilayer is

$$\mathbf{M}_{2N+1} = \mathbf{M}_{2N} \,.\, \mathbf{M}, \tag{97}$$

* A thorough discussion of the properties of a system of double layers will be found in a paper by C. DUFOUR and A. HERPIN, *Rev. Opt.*, **32** (1953), 321.

TABLE III

Reflectivity $\mathscr{R}_{2N+1}$ of multilayers formed by a periodic succession of quarter-wave films of zinc sulphide and cryolite at normal incidence ($n_1 = 1$, $n_2 = 2{\cdot}3$, $n_3 = 1{\cdot}35$, $n_l = 1{\cdot}52$, $n_2h_2 = n_3h_3 = \lambda_0/4$, $\lambda_0 = 5460$ Å, $\theta_1 = 0$).

The values in brackets are experimental results obtained by P. GIACOMO, *Compt. Rend. Acad. Sci., Paris*, **235** (1952), 1627.

N	$\mathscr{R}_{2N+1}$
0	0·306
1	0·672
2	0·872 (0·865)
3	0·954 (0·945)
4	0·984 (0·97)

where $\mathbf{M}_{2N}$ is given by (87) and $\mathbf{M}$ is the characteristic matrix of the last film in the sequence. In particular with quarter-wave films at normal incidence, $\mathbf{M}_{2N}$ reduces to (95), $\beta_2 = \pi/2$, and (97) then becomes

$$\mathbf{M}_{2N+1} = \begin{bmatrix} 0 & -\dfrac{i}{n_2}\left(-\dfrac{n_3}{n_2}\right)^N \\ -in_2\left(-\dfrac{n_2}{n_3}\right)^N & 0 \end{bmatrix}. \tag{98}$$

Substitution into (49) and (51) gives the required reflectivity:

$$\mathscr{R}_{2N+1} = \left(\frac{1-\left(\dfrac{n_2}{n_1}\right)\left(\dfrac{n_2}{n_l}\right)\left(\dfrac{n_2}{n_3}\right)^{2N}}{1+\left(\dfrac{n_2}{n_1}\right)\left(\dfrac{n_2}{n_l}\right)\left(\dfrac{n_2}{n_3}\right)^{2N}}\right)^2. \tag{99}$$

The reflectivity is seen to increase rapidly with the ratio n_2/n_3 and with N (see Table III).