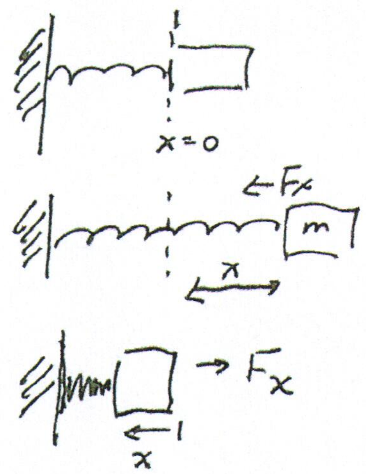
Chap 15 Oscillatory Motion

15.1 Motion of an object attached to a spring

$$F_s = -kx = ma_x$$

$$\therefore a_x = -\frac{k}{m}x$$



1) acceleration is proportional to $\vec{x}$ and ~~object~~ oppositive to the direction of displacement

2) Simple harmonic Motion

Mathematical

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dx^2} = -\frac{k}{m}x$$

let $\frac{k}{m} = \omega^2$ $\qquad dt^2$

$$\frac{d^2x(t)}{dt^2} = -\omega^2 x(t)$$ — 2^{nd} order differential Equation

$$x(t) = A\cos(\omega t + \phi)$$

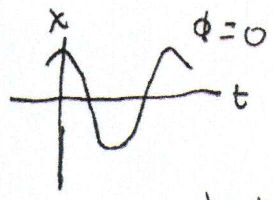
$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 \underbrace{A\cos(\omega t + \phi)}$$
$$= -\omega^2 x$$

$\therefore$ for a Simple harmonic Oscillator

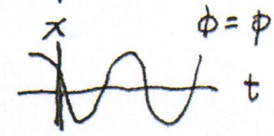
$$\ddot{x}(t) + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$



$$x(t) = A\cos(\omega t + \phi)$$

$\phi \equiv$ phase determined at $t = 0$



$x(t) = x(t+T)$

$[\omega(t+T)+\phi] - (\omega t+\phi) = 2\pi$

$\omega T = 2\pi$ or $T = \frac{2\pi}{\omega}$ $f = \frac{1}{T}$

$= 2\pi\sqrt{\frac{m}{k}}$

$x(t) = A\cos(\omega t+\phi)$

$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi)$, $v_{max} = \pm\omega A$

$a = \frac{dx^2}{dt^2} = -\omega^2 A\cos(\omega t+\phi)$, $a_{max} = \omega^2 A$

① If we pick an initial condition

$t=0$ $x=A$. then $x(0)=A$

$v(0)=0$ initial condition

$$\begin{cases} x(0) = A\cos\phi = A \\ v(0) = -\omega A\sin\phi = 0 \end{cases}$$

$\rightarrow$ if $\phi=0$. $x = A\cos\omega t$ as solution

② If we define $t=0$ @ $x=0$

$x(0) = A\cos\phi = 0$ $\rightarrow \phi = \pm\frac{\pi}{2}$

$v(0) = -\omega A\sin\phi = v_i$ $\rightarrow A = \mp\frac{v_i}{\omega}$

But initial velocity is positive and amplitude must be positive

$\rightarrow \phi = -\frac{\pi}{2}$

$\therefore$ $x(t) = \frac{v_i}{\omega}\cos(\omega t - \frac{\pi}{2})$

do example problems

15.3 Energy of the simple harmonic oscillator

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

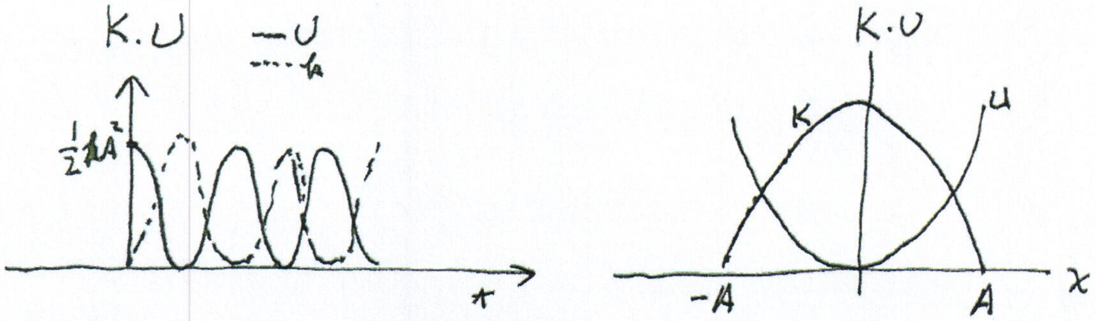
$$U = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$E = K + U = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} m \frac{k}{m} A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2$$

- total energy of a SHO is constant
- $\sim A^2$



Check the similarity of pendulum and SHO
Fig 15.11 (P463)

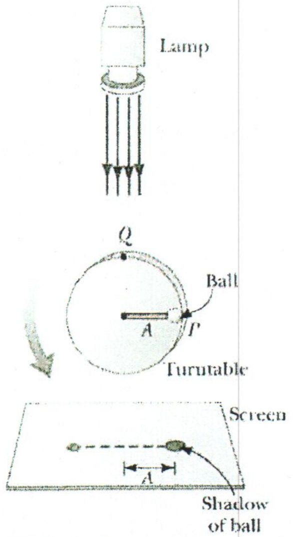
$$E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2$$

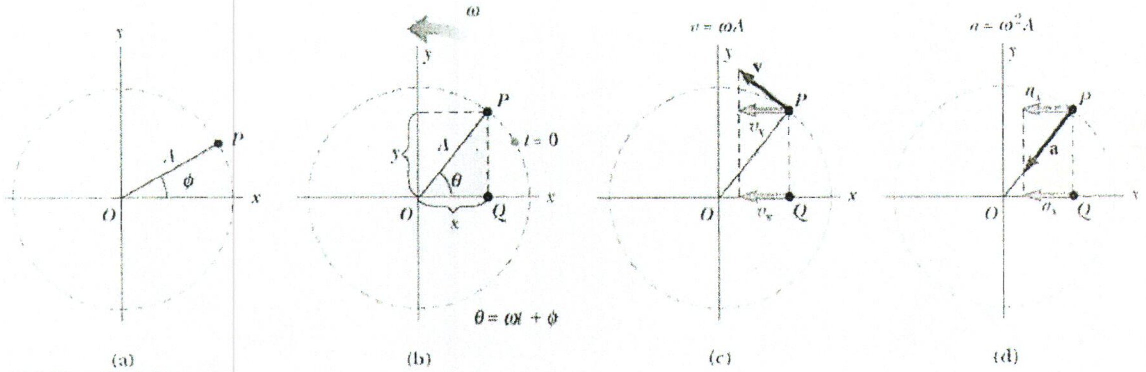
$$\frac{1}{2} m v^2 = \frac{1}{2} k (A^2 - x^2)$$

$$v = \pm \sqrt{\frac{k}{m}(A^2 - x^2)}$$

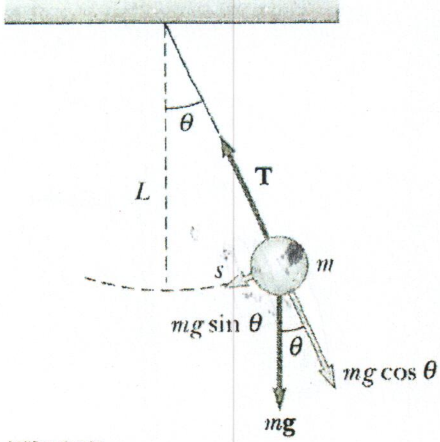
$$= \pm \omega \sqrt{A^2 - x^2}$$

Simple harmonic motion and Uniform Circular Motion





15.5 Pendulum



$$F_t = -mg\sin\theta = ma_t = m\frac{d^2s}{dt^2}$$

But $s = L\theta \quad \frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}$

$$\therefore -mg\sin\theta = mL\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \left(\frac{-g}{L}\right)\sin\theta$$

if $\theta \approx 0 \quad \sin\theta = \theta$

$$\therefore \frac{d^2\theta}{dt^2} = \left(-\frac{g}{L}\right)\theta$$

$\omega = \sqrt{\frac{g}{L}}$ same as that of a SHO

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

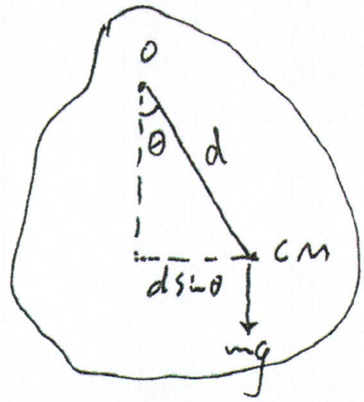
Physical pendulum

$$\tau = I\alpha$$

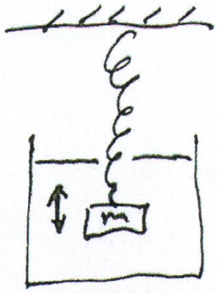
$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{mgd}{I}}$$



15.6 Damped Oscillator

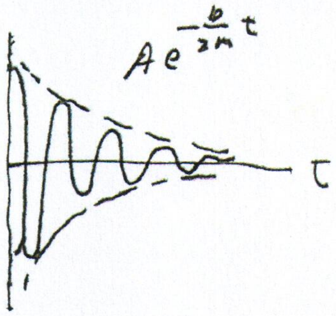


$$\sum F_x = -kx - \underbrace{bV_x}_{\equiv R} = ma_x$$

$$-kx - \underbrace{b\frac{dx}{dt}}_{\text{damping term.}} = m\frac{d^2x}{dt^2}$$

if b is small

$$x(t) = A e^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$



R to compare with bV, kx

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

$$= \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

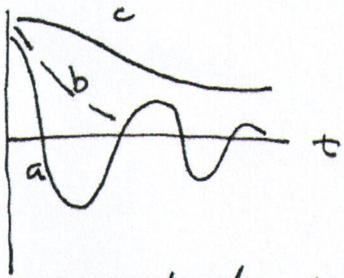
critical damping

$\Rightarrow \omega_0^2 = \left(\frac{b}{2m}\right)^2$

$b = 2m\omega_0$

$|R| = bV_x$

$\omega_0 = \sqrt{\frac{k}{m}}$ — natural frequency



a: $R_{max} < kA$ underdamped

b: $b_c = 2m\omega_0$ Critically damped

c: $R_{max} > kA$ overdamped

$= bV_{max}$

a: underdamped

b = critically damped

c = overdamped oscillator

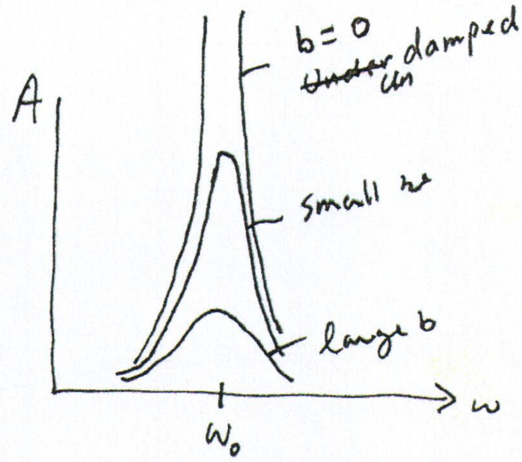
15.7 Forced Oscillation

$$\sum F = ma \qquad \underbrace{F_0 \sin \omega t}_{\substack{\text{driving force} \\ \text{periodic}}} - \underbrace{b \frac{dx}{dt}}_{\text{damped term}} - kx = m \frac{d^2x}{dt^2}$$

$$x = A \cos(\omega t + \phi)$$

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

ω_0 = natural frequency when $b = 0$



When the frequency of a driving force equals that of the natural frequency A increases for undamped Oscillation