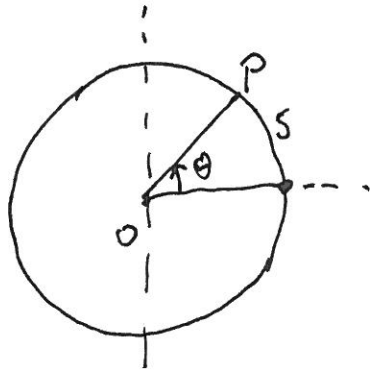


Chap 10. Rotation of a rigid Object about a fixed Axis

P10-1

Rigid \equiv Non deformable

10-1 Angular position, Velocity and Acceleration



$$s = r\theta$$

$$\theta = \frac{s}{r} \text{ [radian]}$$

1 Radian \equiv the angle subtended by an arc length of the radius

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \sim 57.3^\circ$$

$$\Delta\theta \equiv \theta_f - \theta_i$$

Angular speed

Angular speed

$$\bar{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

$$\omega \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular acceleration

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

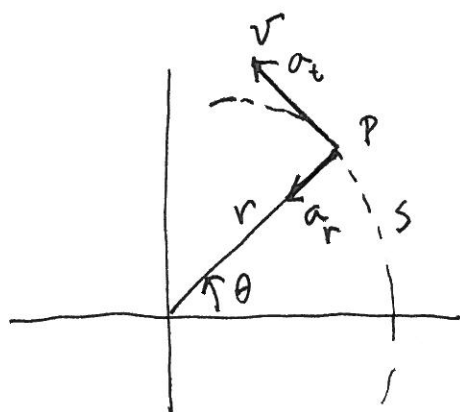
For a rigid object rotates about a fixed axis, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and acceleration.

Use right hand rule to decide the directions

10-2. Rotation Kinematics: Rotational motion with constant angular acceleration

$$\begin{array}{l|l}
 d\omega = \alpha dt & \\
 \omega_f = \omega_i + \alpha t & v_f = v_i + at \\
 \theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2 & x_f = x_i + v_i t + \frac{1}{2} at^2 \\
 \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) & v_f^2 = v_i^2 + 2a(x_f - x_i) \\
 \theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t & x_f = x_i + \frac{1}{2}(v_i + v_f)t
 \end{array}$$

Relating angular and linear quantities



$$v = \frac{ds}{dt} = r \frac{d\theta}{dt}$$

$$\therefore \underline{v = r\omega}$$

$$a_t = \frac{dv}{dt} = r \frac{d\omega}{dt} = r\alpha$$

$$a_r = \frac{v^2}{r} = r\omega^2$$

$$\vec{a} = \vec{a}_t + \vec{a}_r \Rightarrow |\vec{a}| = \sqrt{a_r^2 + a_t^2}$$

$$\begin{aligned}
 |\vec{a}| &= \sqrt{r^2\alpha^2 + r^2\omega^4} \\
 &= r\sqrt{\alpha^2 + \omega^2}
 \end{aligned}$$

10.4 Rotational Kinetic energy

$$K_i = \frac{1}{2} m_i v_i^2 \quad \text{for } i\text{th point with mass } m_i$$

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$= \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

$$I \equiv \sum_i m_i r_i^2$$

= moment of inertia

$$= \frac{1}{2} I \omega^2$$

- Rotational Kinetic energy.

10.5 Calculating moments of inertia

P10-3

$$I = \sum_i r_i^2 m_i$$

$$= \lim_{\Delta m_i \rightarrow 0} \sum_i r_i^2 \Delta m_i$$

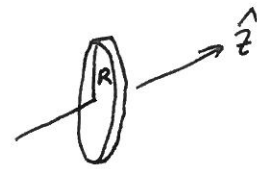
$$= \int r^2 dm \quad \text{But } dm = \rho dV$$

$$I = \int \rho r^2 dV$$

Example of the moment of inertia

(a) Hoop about central axis

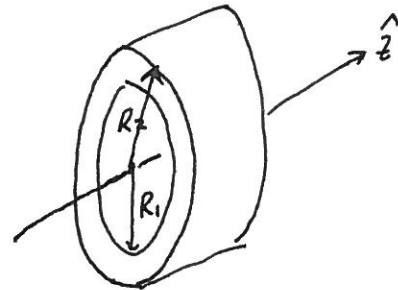
$$I = \int_0^M R^2 dm = MR^2$$



(b) Annular cylinder

$$I = \int dm r^2$$

$$dm = 2\pi r H \rho dr$$



$$= \int 2\pi r H \rho dr r^2$$

$$= 2\pi H \rho \int_{R_1}^{R_2} r^3 dr = 2\pi H \rho \frac{1}{4} (R_2^4 - R_1^4)$$

$$\text{But } M = \pi R_2^2 H \rho - \pi R_1^2 H \rho$$

$$= \pi H \rho (R_2^2 - R_1^2)$$

$$\therefore I = \frac{\pi}{2} H \rho (R_2^4 - R_1^4) = \frac{\pi}{2} H \rho (R_2^2 + R_1^2) (R_2^2 - R_1^2)$$

$$= \frac{1}{2} M (R_1^2 + R_2^2)$$

(c) Solid cylinder

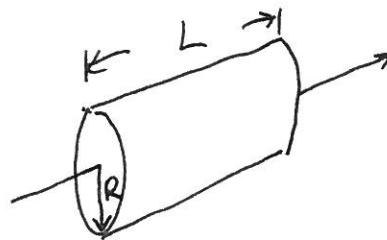
$$I = \int r^2 dm$$

$$= \int r^2 2\pi r dr L \rho$$

$$= 2\pi L \rho \int_0^R r^3 dr$$

$$= 2\pi L \rho \frac{R^4}{4}$$

$$= \frac{1}{2} M R^2$$



(d) Thin rod about axis perpendicular to the length

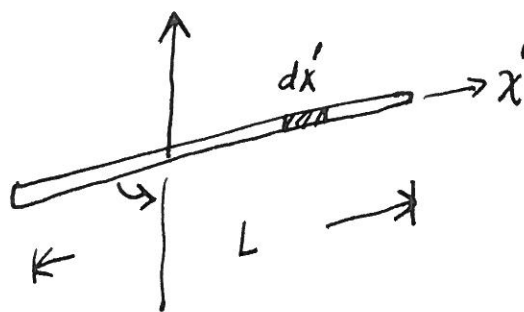
$$dm = \lambda dx' = \frac{M}{L} dx'$$

$$I = \int r^2 dm = \int_{-\frac{L}{2}}^{+\frac{L}{2}} \lambda (x')^2 dx'$$

$$= \frac{M}{L} \int_{-\frac{L}{2}}^{+\frac{L}{2}} (x')^2 dx'$$

$$= \frac{M}{L} \left[\frac{(x')^3}{3} \right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{1}{12} ML^2$$



(e) Solid sphere about any diameter
cut the sphere into many horizontal
disks of radius r .

$$dI_{\text{disk}} = \frac{1}{2} r^2 dm$$

$$dI_{\text{disk}} = \frac{1}{2} r^2 dm = \frac{1}{2} (R^2 - x^2) \pi \rho (R^2 - x^2) dx$$

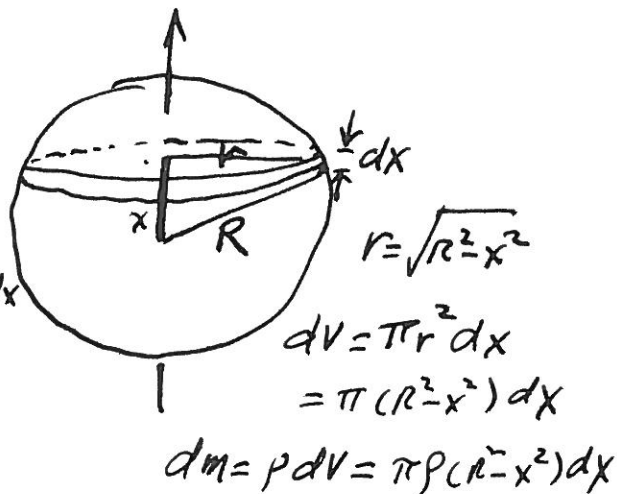
$$= \frac{\pi \rho}{2} (R^2 - x^2)^2 dx$$

$$I = (2) \frac{\pi \rho}{2} \int_0^R (R^2 - x^2) dx$$

$$= \frac{8\pi \rho R^5}{15}$$

$$\text{But } \rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

$$\therefore I = \left(\frac{8\pi R^5}{15} \right) \left(\frac{3M}{4\pi R^3} \right) = \frac{2}{5} MR^2$$

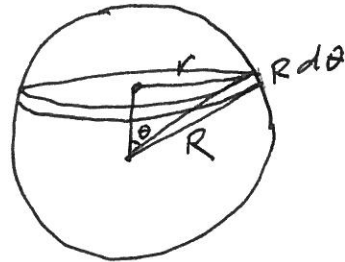


Note: There are many different ways to calculate the moment of inertia; this is just one way.

(g) Thin spherical shell about any diameter

P10-6

$$\begin{aligned}
 I &= \int_0^\pi r^2 dm \\
 &= \int_0^\pi R^2 \sin^2 \theta \cdot 2\pi R^2 \sin \theta d\theta \\
 &= 2\pi R^4 \int_0^\pi \sin^2 \theta \sin \theta d\theta \\
 &= 2\pi R^4 \int_{-1}^1 (1 - \cos^2 \theta) d(-\cos \theta) \\
 &= 2\pi R^4 \int_{-1}^1 (1 - \cos^2 \theta) d(\cos \theta) \\
 &= 2\pi R^4 \left[x - \frac{1}{3} x^3 \right]_{-1}^1 \\
 &= 2\pi R^4 \cdot \frac{4}{3} \\
 &= \frac{8}{3} \pi R^4 = \frac{2}{3} MR^2
 \end{aligned}$$



$$\begin{aligned}
 r &= R \sin \theta \\
 dm &= 2\pi r R d\theta \\
 &= 2\pi R \sin \theta R d\theta \\
 &= 2\pi R^2 \sin \theta d\theta \\
 M &= 4\pi R^2
 \end{aligned}$$

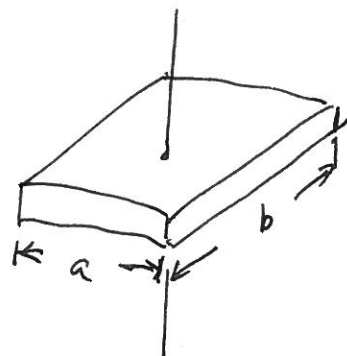
(h) Hoop about any diameter

$$\begin{aligned}
 I &= \int R^2 dm = 2 \int_0^\pi (R \sin \theta)^2 dm \\
 &= 2 \int_0^\pi R^2 \sin^2 \theta \cdot R d\theta = 2R^3 \int_0^\pi \sin^2 \theta d\theta \\
 &= \frac{1}{2} MR^2 \quad M = 2\pi R \\
 &= \frac{1}{2} MR^2
 \end{aligned}$$

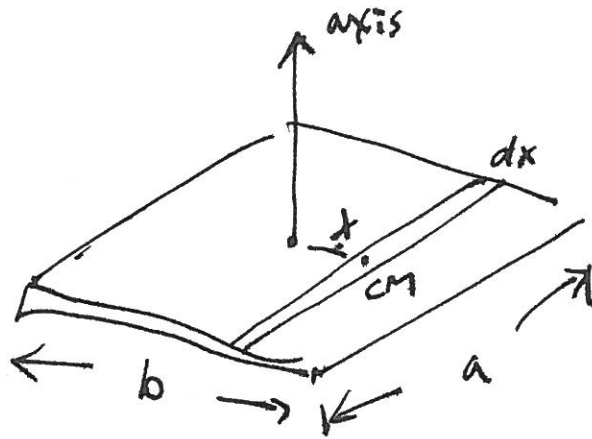


(i) Slab about perpendicular axis through center

$$\begin{aligned}
 I &= \int r^2 dm = I_x + I_y \\
 &= \int (x^2 + y^2) dm \\
 &= \int x^2 dm + \int y^2 dm \\
 &= \frac{1}{12} (a^2 + b^2)
 \end{aligned}$$



Also see page P10-6-1



$$dm = a dx$$

$$\frac{dM}{M} = \frac{a dx}{ab} \Rightarrow dm = \frac{M}{b} dx$$

$$dI \text{ of the strip} = dI_{cm} + dm h^2$$

$$= \frac{1}{12} (dm) a^2 + dm x^2$$

$$= \frac{1}{12} \left(\frac{M}{b}\right) dx a^2 + \left(\frac{M}{b}\right) dx x^2$$

$$= \frac{Ma^2}{12b} dx + \frac{M}{b} x^2 dx$$

$$I = \int dI = \frac{Ma^2}{12b} \int_{-\frac{b}{2}}^{\frac{b}{2}} dx + \frac{M}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} x^2 dx$$

$$= \frac{1}{12} M (a^2 + b^2)$$

Parallel axis theorem

P10-7

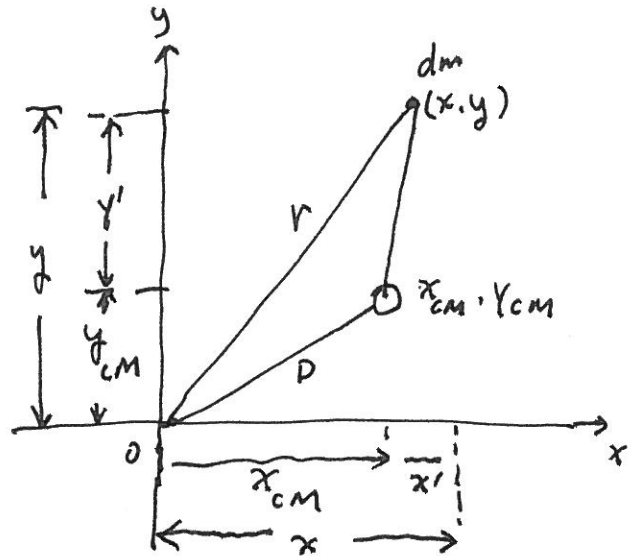
$$I = I_{cm} + MD^2$$

$$I = \int r^2 dm$$

$$= \int (x^2 + y^2) dm$$

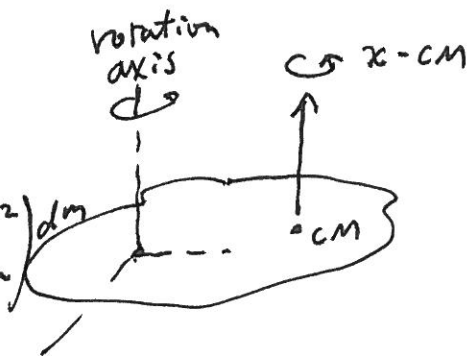
But $x = x' + x_{cm}$

$$y = y' + y_{cm}$$



$$\therefore I = \int [(x' + x_{cm})^2 + (y' + y_{cm})^2] dm$$

$$= \int (x'^2 + 2x'x_{cm} + x_{cm}^2 + y'^2 + 2y'y_{cm} + y_{cm}^2) dm$$



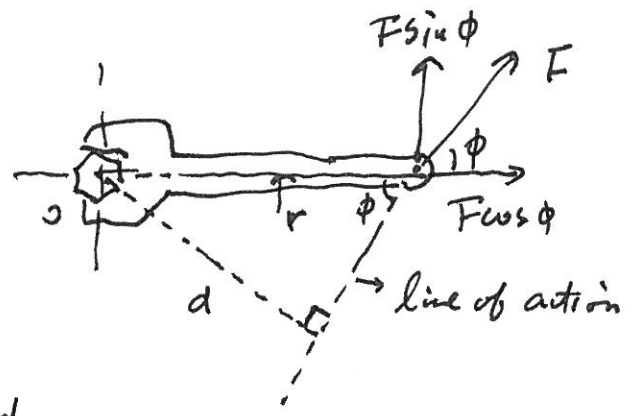
$$= \underbrace{\int (x'^2 + y'^2) dm}_{I_{cm}} + \underbrace{2x_{cm} \int x' dm}_{=0} + \underbrace{2y_{cm} \int y' dm}_{=0} + \underbrace{(x_{cm}^2 + y_{cm}^2) \int dm}_{D^2 M}$$

definition of center of mass

$$\therefore I = I_{cm} + MD^2$$

← Parallel axis theorem

10.6 Torque

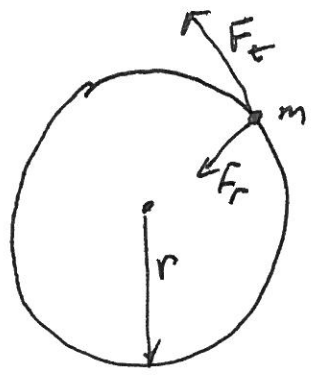


torque $\equiv \tau$
 $= r F \sin \phi = F d$

$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 + F_2 d_2$
 - for more than one torques.

10.7 Relationship between torque and Angular acceleration

$F_t = m a_t$
 $\tau = F_t r = m a_t r$
 But $a_t = r \alpha$



$\therefore \tau = m (r \alpha) r$
 $= m r^2 \alpha$

$\tau = I \alpha$

$\therefore \tau \propto \alpha$

Torque is proportional to its Angular acceleration, and the proportionality constant is the moment of inertia

Extend it to rigid object of arbitrary shape

$dF_t = dm a_t$
 $d\tau = r dF_t = a_t r dm = r \alpha r dm = r^2 \alpha dm$

$\sum \tau = \int d\tau = \int \alpha r^2 dm = \alpha \int r^2 dm$

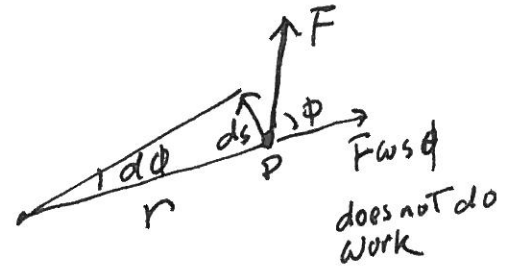
$\therefore \sum \tau = I \alpha$

10.8. Work, Power and Energy in rotational motions

P10-9

$$dW = F \cdot ds = F \sin \phi \underbrace{r d\theta}_{\tau d\theta}$$

$$\therefore \underline{dW = \tau d\theta}$$



$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} \quad \therefore \underline{P = \frac{dW}{dt} = \tau \omega}$$

$$\sum \tau = I \alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\theta} \frac{d\theta}{dt} = I \frac{d\omega}{d\theta} \omega$$

$$\sum \tau d\theta = dW = I \omega d\omega$$

$$\sum W = \int_{\omega_i}^{\omega_f} I \omega d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

Work - Kinetic energy theorem
for rotational motions

10.9 Rolling motion of a rigid object

① The center of mass moves

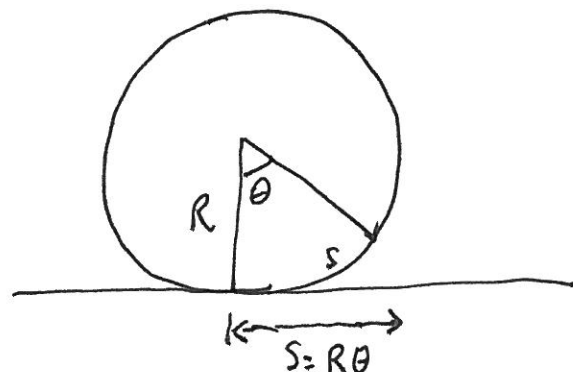
$$s = R\theta$$

Fig 10-27, P. 317

② The linear speed of CM is

$$v_{CM} = \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega$$

for a cylinder or sphere
rolls without slipping



→ Condition for pure rolling

$$a_{cm} = \frac{dV_{cm}}{dt} = R \frac{d\omega}{dt} = R\alpha$$

Check also Fig 10.28 (p 317)

The total Kinetic energy of the rolling cylinder.

$$K = \frac{1}{2} I_P \omega^2 \quad I_P \equiv \text{moment of inertia about point P.}$$

$$= \frac{1}{2} (I_{cm} + MR^2) \omega^2$$

$$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MR^2 \omega^2$$

$$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M \underbrace{V_{cm}^2}$$

Rotation about
center of mass

translation of center of mass

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M V_{cm}^2$$

$$= \frac{1}{2} I_{cm} \left(\frac{V_{cm}^2}{R^2} \right) + \frac{1}{2} M V_{cm}^2$$

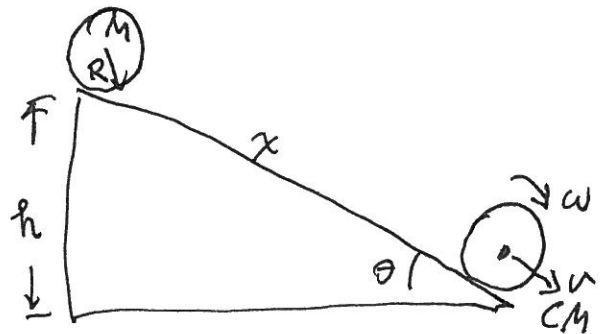
$$= \frac{1}{2} \left(\frac{I_{cm}}{R^2} + M \right) V_{cm}^2$$

, But $V_{cm} = R\omega$

$$K_f + U_f = K_i + U_i$$

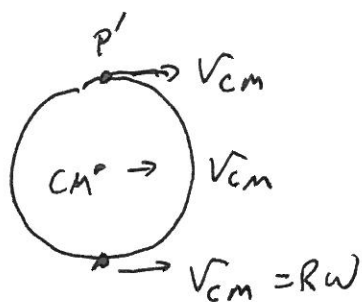
$$\frac{1}{2} \left(\frac{I_{cm}}{R^2} + M \right) V_{cm}^2 + 0 = 0 + Mgh$$

$$V_{cm} = \left(\frac{2gh}{1 + \frac{I_{cm}}{MR^2}} \right)^{\frac{1}{2}}$$

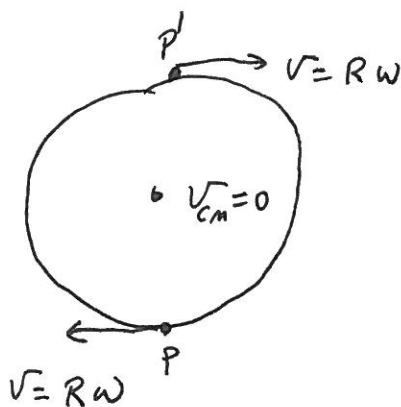


Page 318, Fig 10.29

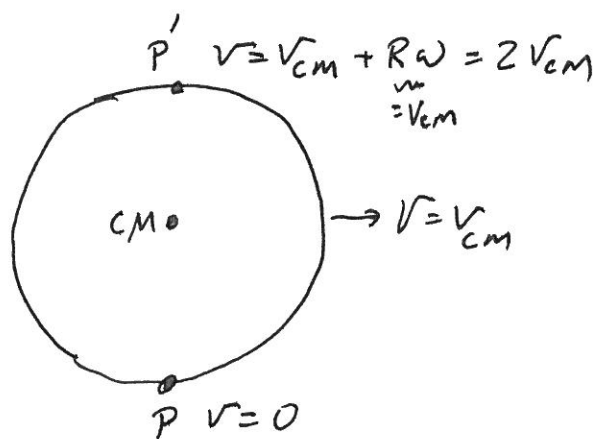
(a) pure translation



(b) pure rotation



(c) Combination of translation + rotation



a combination of (a) + (b)