Here is sketches of proofs for Homework problems #2, #3 in §3.▯.

EX 1 (§3.8 #2) Let X be a random variable with discrete uniform pmf on $\{\pm 3, \pm 2, \pm 1, 0\}$. Let g be pmf of the random variable $Y = r(X) = X^2 - X$. Calculate the pmf g for Y.
Solution. Note that for this question, it might be easier to follow the definition (1) then trying to figure out the inverse of r over various subdomains.

First we will make clear what the values that Y can take. Specifically, let

$$X = \{x|f(x) > 0\} = \{\pm 3, \pm 2, \pm 1, 0\}$$

We would like to get

$$Y = \{y|g(y) > 0\} = \{y|y = r(x), x \quad X\}.$$

Note

x	-3	-2	-1	0	1	2	3
y	12	6	2				

where $1[A]$ is the indicator function of event A. Since

$$\begin{aligned} r(x) = x(2-x) \le y &\Leftrightarrow x^2 - 2x + y \ge 0 \\ &\Leftrightarrow (x \ge 1 + \sqrt{1-y}) \text{ or } (x \le 1 - \sqrt{1-y}). \end{aligned}$$

Note for $y = r(x) = x(2-x)$, the derivative $y' = -(x-1)$. Thus for $x \in [0,2]$ (including 0, 2 makes no differences since f is a pdf), $y = r(x)$ is increasing for $x < 1$; decreasing for $x > 1$. In addition, it reaches its maximum 1 when $x = 1$ and attains its minimum 0 at $x = 0, 2$. Hence $G(y) = 0$ if $y < 0$; $= 1$ if $y > 1$; for $y \in [0,1]$,

$$\begin{aligned} (2) &= \int_0^2 1[(x \ge 1 + \sqrt{1-y}) \text{ or } (x \le 1 - \sqrt{1-y})] \frac{x}{2}\, dx \\ &= \left(\int_{1+\sqrt{1-y}}^2 + \int_0^{1-\sqrt{1-y}}\right) \frac{x}{2}\, dx = \frac{x^2}{4}\Big|_{1+\sqrt{1-y}}^2 + \frac{x^2}{4}\Big|_0^{1-\sqrt{1-y}} \\ &= 1 - \frac{1}{4}[(1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2] = 1 - \sqrt{1-y}. \end{aligned}$$

To summarize,

$$G(y) = \begin{cases} 0, & y < 0 \\ 1 - \sqrt{1-y}, & y \in [0,1] \\ 1, & y > 1. \end{cases}$$

Thus the pdf

$$g(y) = \frac{d}{dy} G(y) = \begin{cases} \frac{1}{2\sqrt{1-y}}, & y \in (0,1), \\ 0, & \text{Otherwise.} \end{cases}$$

♠

Remark 1 A way to check whether we do get the right pdf/pmf is to calculate the expectation of Y (if exists) in two ways and confirm the equivalence. Specifically, for the case when X has pdf f and $Y = r(X)$ has pdf g

$$E(Y) = \int y g(y)\, dy \tag{3}$$

$$E(r(X)) = \int r(x) f(x)\, dx. \tag{4}$$

§3.□: #3 as an illustration.

$$E(r(X)) = EX(2-X) = \int_0^2 2x - x^2 \, dx = \frac{2}{3}.$$

$$\begin{aligned}
E(Y) &= \int_0^1 y \frac{1}{2\sqrt{1-y}} \, dy, \qquad \text{let } y = \sin^2\theta, \theta \in (0, \pi/2) \\
&= \int_0^{\pi/2} \frac{\sin^2\theta}{2\sqrt{1-\sin^2\theta}} 2\sin\theta\cos\theta \, d\theta \\
&= \int_0^{\pi/2} \sin^3\theta \, d\theta = \int_{\theta=0}^{\theta=\pi/2} (\cos^2\theta - 1) \, d\cos\theta \\
&= \left[\frac{z^3}{3} - z\right]_1^0 = \frac{2}{3}.
\end{aligned}$$

This confirms that we did get the right pdf! The same idea works for moment generating functions as well. It might be easier to calculate in some cases. ♠

Finally, I supplement a proof of a claim made in the Textbook. This proof is for the curious enjoyment only. :)

EX 3 (Example 4.4.1, P 204)

$$\int_{-\infty}^{\infty} |x|^k e^{-(x-3)^2} \, dx < \infty \qquad \forall k \in N.$$

Proof. For any positive integer k, by L'Hôpital's Rule

$$|x|^k e^{-\frac{(x-3)^2}{2}} \to 0.$$

Thus for any $\epsilon > 0$, there exists an $M(\epsilon) > 0$ such that

$$|x|^k e^{-\frac{(x-3)^2}{2}} < \epsilon, \qquad \forall |x| > M.$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} |x|^k e^{-(x-3)^2} \, dx &= 2\int_0^{\infty} x^k e^{-(x-3)^2} \, dx \\
&= 2\left(\int_0^M + \int_M^{\infty}\right) x^k e^{-(x-3)^2} \, dx \\
&\le 2\left(\int_0^M x^k e^{-(x-3)^2} \, dx + \epsilon \int_M^{\infty} e^{-\frac{(x-3)^2}{2}} \, dx\right) \\
&\le 2\left(M^k + \int_M^{\infty} e^{-(x-3)^2} \, dx + \epsilon \int_M^{\infty} e^{-\frac{(x-3)^2}{2}} \, dx\right) < \infty.
\end{aligned}$$

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You can refer to P 277, Example 1, Mathematical Analysis, 2nd Edition, Apostol for more