One of the homework problems illustrates that calculating mean and variance of a random vector sometimes is easier than dealing with random variables individually using those long summation notations.

1. Under G-M. condition, define $e_i = Y_i - \hat{Y}_i$. Show that

$$Var(e_i) = \sigma^2 - Var(\hat{Y}_i), \quad i = 1, \cdots, n.$$

Actually, (5.75) in our text is exactly what we are after. However, we don't really need to assume $(X\ X)^{-1}$ exists nor assume the simple linear model. Some of your proofs are close but not quite right. Here I pretty much follow the proof given by Ms. Chang, J.-L. with a little modification.

Proof. By definition, $e = Y - \hat{Y} = (I - P)Y$ where P is a projection matrix onto V_r, the column space of X, such that $\hat{Y} = PY$. Note that since P is a projection matrix, so is $I - P$ where I is the nxn identity matrix. And for any projection (idempotent) matrix, Q, $Q^2 = Q$ and $Q' = Q$.
Therefore

$$\begin{aligned}\Sigma_e &= \Sigma_{(I-P)Y} = (I-P)\Sigma_Y(I-P) \\ &= (I-P)\sigma^2(I-P) = \sigma^2(I-P) \\ &= \sigma^2 I - P\sigma^2 P = \sigma^2 I - \Sigma_{PY} \\ &= \sigma^2 I - \Sigma_{\hat{Y}.}\end{aligned}$$

Hence the diagonal entries of Σ_e equal to those of $\sigma^2 I - \Sigma_{\hat{Y},}$ i.e.

$$Var(e_i) = \sigma^2 - Var(\hat{Y}_i), \quad i = 1, \cdots, n.$$

$\surd$

For those who are rusty on matrix algebra or unfamiliar with random vector manipulation, please referred to §5.7–§5.9 of our textbook. For those are really interested in this topic or need them in your research, you can find more in Searle (1982). Practice them for a while, you will get to love them. :)

References

Searle, S. R. (1982).
Matrix Algebra Useful for Statistics. Wiley.