Note: The exam has 6 questions, for a total of 105 points. Explain your answer and write down the necessary details of your calculation. No explanation/details = No credits. *Good Luck!*

Some definitions and function values for your reference:

- $\Phi(x) = P[Z \le x] = \int_{-\infty}^{x} \phi(t)dt$, cdf of a standard normal r.v. $Z \sim N(0,1)$.
- $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, pdf of a standard normal r.v.
- z_α satisfies $\Phi(z_\alpha) = 1 - \alpha$.
- $\Phi(1.96) = 0.975$, $\Phi(1.645) = 0.95$, $\Phi(1.282) = .90$.

1. *True or False?* In the following questions, determine whether the statement is true or false. Give an example if the statement is true; otherwise, give a counterexample or a brief explanation.

(5) (a) Let X be a discrete random variable with mean μ and variance σ^2. Then $\frac{X-\mu}{\sigma}$ has mean 0 and its variance is 1.

(5) (b) Let $X_1, \cdots, X_n$ be iid (independently identically distributed) random variable with mean μ and variance σ^2. Then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, i.e. a normal with mean 0 and variance 1 where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(5) (c) In the hypothesis testing problem, for a given significance level α, say 0.05, if the data can not lead to the rejection of the null hypothesis H_0 then it implies that there is strong evidence suggesting H_0 is true.

(5) (d) Let $X_1, \cdots, X_n$ iid from a distribution with finite mean and variance. If n is sufficiently large then, by Central Limit Theorem, we can approximate this distribution by a normal distribution.

2. Let $X_1, \cdots, X_n$ be iid random sample from a pdf or pmf f with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ for all i. For construct a confidence interval for μ.

(5) (a) Give suitable assumptions/conditions such that

$$C_1(\mathbf{X}) = [\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$$

is a $(1-\alpha)$ confidence interval for μ.

(5) (b) Verify, under the condition you given above, that $C_1(\mathbf{X})$ is indeed an *exact* $(1-\alpha)$ confidence interval for μ.

(5) (c) Give suitable assumptions/conditions such that $C_1(\mathbf{X})$ is a $(1-\alpha)$ *approximate* confidence interval for μ.

(5) (d) Verify, under the condition you given above, that $C_1(\mathbf{X})$ is indeed an approximate $(1-\alpha)$ confidence interval for μ.

(10) (e) Comment on the difference between a computed 90% CI for μ and a 90% CI for μ. Which one has 90% probability covering the true μ? What is the probability covering the true μ for the other interval?

3. Assume that IQ scores for students in math department in NXX university are approximately $N(\mu, 100)$. The chair of math department claims that the students in the department actually have mean IQ greater than 110.

(10) (a) Formulate a hypothesis testing problem to prove this claim. State your null hypothesis H_0 and alternative hypothesis H_1 clearly and explain briefly why you choose them so.

(10) (b) Assume a random IQ sample of size $n = 16$ is drawn from the students in the department with the sample average $\bar{x} = 115$. Do you reject or accept H_0 at 0.05 significant level? What is your conclusion?

(10) (c) Give a rough estimate of the corresponding p-value (Use the suitable distribution table in the Appendix).

(10) 4. Let $X_1, \cdots, X_n \sim_{iid}$ Bernoulli(p). Show that the length of an approximate $(1-\alpha)$ confidence interval for p is no longer than 2ϵ if n is greater than $\frac{z^2_{\alpha/2}}{4\epsilon^2}$.

(10) 5. Give an illustration of the difference of the paired design and the two-sample (population) design for testing the difference of two normal means. How do the difference in the designs affect the hypothesis tests?

(5) 6. Estimate the points you will get for the exam (excluding this problem).