Note: The exam has 6 questions, for a total of 115 points. Explain your answer and write down necessary details of your calculation. No explanation/details = No credits. *Good Luck!*

1. *True or False?* In the following questions, determine whether the statement is true or false. Justify/Explain your answer. Give an example or explain briefly if the answer is "True", otherwise, give a counterexample.
   (5) (a) Let A, B be two events and $P(A), P(B) > 0$. If A, B are independent then $A \cap B \neq \emptyset$.
   (5) (b) If X is a continuous random variable with pdf (probability density function) f then $P(X = x) > 0$ for some x.
   (5) (c) Let X and Y are random variables and $E(X^2), E(X), E(Y) < \infty$ then $E(X^2 - 2X - Y) = E(X^2) - 2E(X) - E(Y)$.
   (5) (d) Let X and Y are independent random variables and $Var(X), Var(Y) < \infty$ then $Var(X - 2Y + 4) = Var(X) + 4Var(Y)$
   (5) (e) Let Z be a a standard normal random variable (a normal random variable with mean 0 and variance 1) and let $g(x) = P(Z \le x)$. Then $g(Z)$ is distributed as $U(0, 1)$.

(20) 2. Let $X \sim Poisson(\lambda)$ with pdf

$$f(x) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!} & \text{if } x = 0, 1, 2, \ldots; \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $E(X^2)$ and $E(X^3)$.

3. Let $X, Y \sim N(0, 1)$ and X, Y are independent. Let $T = X + Y,\ W = X - Y$.
   (10) (a) Calculate $P(max(X, Y) > 0)$
   (10) (b) Find the joint pdf of T, W.
   (10) (c) Calculate $P(T < 0|W < 0)$.

4. Let $X \sim U(0, 1)$ and define $Y = 1I_{B_1}(X) + 2I_{B_2}(X)$ where $I_A(x)$, the indicator function of event A, takes value 1 if $x \in A$; 0, otherwise. Let $B_1 = (0.1, 0.2), B_2 = (0.3, 0.5)$.
   (10) (a) Determine the pdf of Y.
   (5) (b) Verify that the pdf of Y you derived above is indeed a pdf.

5. Let $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where $\mu_1, \mu_2 \in R$, $\sigma_1^2, \sigma_2^2 > 0$ and $|\rho| < 1$. It is known that (you may use them as given)

$$X_2|X_1 = x_1 \sim N(b(x_1), \sigma_2^2(1 - \rho^2))$$

with $b(x_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1)$ and marginally $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$.
   (10) (a) Derive $Var(E(X_2|X_1))$.
   (10) (b) Assume it is known that $\sigma_1 = \sigma_2 = 1$. Find a number $1 > c > 0$ such that $P(|X_1 - \mu_1| > 2) \le c$. **Hint**: Chebyshev's inequality.

(5) 6. Estimate the points you will get in the exam (excluding this problem).