Note: The exam has 8 questions, for a total of 105 points. Explain your answer and write down necessary details of your calculation. No explanation/details = No credits. *Good Luck!*

You can use these facts if necessary: Let $Y \sim \chi^2_m$, a chi-square random variable with degree of freedom m. Then $E(Y) = m$ and $Var(Y) = 2m$. Furthermore, if $Y_1, \cdots, Y_n \sim_{iid} \chi^2_m$ then $\sum_{i=1}^n Y_i \sim \chi^2_{mn}$.

1. *True or False?* In the following questions, determine whether the statement is true or false. Justify/Explain your answer. Give an example or explain briefly if the answer is "True", otherwise, give a counterexample.

   Let $X_1, X_2, ..., X_n \sim_{iid} f_\theta(x)$ with $\theta \in \mathcal{R} = (-\infty, \infty)$ and $X = (X_1, \cdots, X_n)'$.

(5) (a) If a statistic $T(X)$ is sufficient for θ then $(T(X))^4$ is also sufficient for θ.

(5) (b) If $\delta(X)$ is a uniformly minimum variance unbiased estimator (UMVUE) for θ then $\delta(X)$ has the smallest mean square error among all estimators.

(5) (c) If $T(X)$ is a maximum likelihood estimator (MLE) for θ then $T^2(X)$ is a MLE for θ^2.

(5) (d) If $S(X)$ is unbiased for θ then $S(X)$ is sufficient for θ.

(10) 2. Let X has a discrete pdf $f(x|\theta), \theta \in \Theta = \{0, 1, 2\}$ given below (For example, $f(1|0) = 0.3, f(-1|1) = 0.4$.)

	θ		
x	0	1	2
1	0.3	0.5	0.2
0	0.5	0.2	0.4
-1	0.2	0.3	0.2

Find MLE of θ^3. Is this MLE (Maximum Likelihood Estimator) unbiased?

(10) 3. Let $X \sim U(0, 1)$ and define $Y = I_{B_1}(X) + I_{B_2}(X)$ where $I_A(x)$, the indicator function of event A, takes value 1 if $x \in A$; 0, otherwise. Let $B_1 = (0.1, 0.3), B_2 = (0.3, 0.6)$. Determine the pdf of Y.

4. Let $X, \cdots, X_n \sim_{iid} U(-\theta, \theta)$ with $\theta > 0$ and its pdf

$$f_\theta(x) = \begin{cases} \frac{1}{2\theta} & \text{if } -\theta \leq x \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

(10) (a) Find a non-trivial sufficient statistic for θ.

(10) (b) Find MLE for θ.

5. Let $X_1, \cdots, X_n \sim_{iid}$ Poisson(λ) with $\theta > 0$ and its pdf

$$f_\lambda(x) = \begin{cases} e^{-\lambda}\ \frac{\lambda^x}{x!}, & \text{if } x = 0, 1, \cdots; \\ 0 & \text{otherwise.} \end{cases}$$

Recall $E(X_i) = Var(X_i) = \lambda$, for all $i = 1, \cdots, n$.

(10) (a) Find a complete (non-trivial) sufficient statistic $T(X)$ for λ.

(10) (b) Compute $\delta(X) = E(X_1|T(X))$. Is it a UMVUE for λ? Why?

(10) 6. Let $X_1, \ldots, X_n$ be *iid* sample from $N(0, \sigma^2), \sigma^2 > 0$ and its pdf

$$f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}}, \quad x \in \mathcal{R}.$$

Derive the Cramér-Rao lower bound for estimating σ^2 and find a UMVUE for σ^2.

(10) 7. Let $X_1, \cdots, X_n$ be iid random variables with mean μ and variance σ^2, both unknown. Show that $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is a minimum variance unbiased linear estimator of μ.

(5) 8. Estimate the points you will get in the exam (excluding this problem).