Note: The exam has 6 questions, for a total of 115 points. Explain your answer and write down necessary details of your calculation. No explanation/details = No credits. *Good Luck!*

1. *True or False?* In the following questions, determine whether the statement is true or false. Justify/Explain your answer. Give an example or explain briefly if the answer is "True", otherwise, give a counterexample.

(5) (a) If X is a continuous random variable with pdf (probability density function) f then $P(X = x) = 0$ for some x.

(5) (b) If X, Y are independent and $E(\frac{X}{Y^2}) = E(X)/E(Y^2)$ provided all these expectations exist.

(5) (c) Let X and Y are independent random variables and $Var(X), Var(Y) < \infty$ then $Var(X - 2Y + 4) = Var(X) + 4Var(Y) - 2Cov(X, Y)$.

Let $X_1, X_2, ..., X_n \sim_{iid} f_\theta(x)$ with $\theta \in \mathcal{R} = (-\infty, \infty)$ and $X = (X_1, \cdots, X_n)'$.

(5) (d) If a statistic $T(X)$ is sufficient for θ then $(T(X))^3$ is also sufficient for θ.

(5) (e) If $\delta(X)$ is a uniformly minimum variance unbiased estimator (UMVUE) for θ and $Var(\delta(X))$ then $\delta(X)$ has the smallest mean square error among all unbiased estimators.

(5) (f) If $T(X)$ is a maximum likelihood estimator (MLE) for θ then $T^2(X)$ is a MLE for θ.

(5) (g) If $S(X)$ is unbiased for θ then $S^2(X)$ is unbiased for θ^2.

(15) 2. Let X has a discrete pdf $f(x|\theta), \theta \in \Theta = \{0, 1\}$ given below
(For example, $f(1|0) = 0.3, f(-1|1) = 0.4$.)

	θ	
x	0	1
1	0.3	0.4
0	0.5	0.2
-1	0.2	0.4

Find MLE of θ^2. Is this MLE (Maximum Likelihood Estimator) unbiased?

(10) 3. Let $X, \cdots, X_n$ iid with pdf

$$f_\theta(x) = \begin{cases} \frac{2}{\theta^2}(\theta - x) & \text{if } 0 < x < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

for all $\theta \in \Theta = (0, \infty)$. Find a non-trivial sufficient statistic for θ if possible.

4. Let $X_1, \cdots, X_n \sim_{iid} Poisson(\lambda)$ with $\lambda > 0$ and its pdf

$$f_\lambda(x) = \begin{cases} e^{-\lambda} \frac{\lambda^x}{x!}, & \text{if } x = 0, 1, \cdots; \\ 0 & \text{otherwise.} \end{cases}$$

Recall $E(X_i) = Var(X_i) = \lambda$, for all $i = 1, \cdots, n$.

(10) (a) Find a complete sufficient statistic for λ.

(20) (b) Find a UMVUE for λ. Is Cramér-Rao lower bound (CRLB) attainable in this case?

5. Let $X_1, \ldots, X_n$ be *iid* sample from $N(\mu, 1), \mu \in \mathcal{R}$ and its pdf

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{(x-\mu)^2/2}, \quad x \in \mathcal{R}.$$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$.

(10) (a) Show that the statistic $T(X) = \bar{X}$ is sufficient for μ and is complete.

(10) (b) Derive the CRLB for estimating μ and find UMVUE for μ.

(5) 6. Estimate the points you will get in the exam (excluding this problem).