The transformed equation is $(3+\sqrt{2})u^2 + (3-\sqrt{2})v^2 = 4$, which represents an ellipse with semi-major axis $2/\sqrt{3-\sqrt{2}}$ and semi-minor axis $2/\sqrt{3+\sqrt{2}}$. (We will discover another way to do a question like this in Section 13.3.)

Exercises 8.1

Find equations of the conics specified in Exercises 1–6.

1. ellipse with foci at $(0, \pm 2)$ and semi-major axis 3.
2. ellipse with foci at $(0, 1)$ and $(4, 1)$ and eccentricity $1/2$.
3. parabola with focus at $(2, 3)$ and vertex at $(2, 4)$.
4. parabola passing through the origin and having focus at $(0, -1)$ and axis along $y = -1$.
5. hyperbola with foci at $(0, \pm 2)$ and semi-transverse axis 1.
6. hyperbola with foci at $(\pm 5, 1)$ and asymptotes $x = \pm(y - 1)$.

In Exercises 7–15, identify and sketch the set of points in the plane satisfying the given equation. Specify the asymptotes of any hyperbolas.

7. $x^2 + y^2 + 2x = -1$
8. $x^2 + 4y^2 - 4y = 0$
9. $4x^2 + y^2 - 4y = 0$
10. $4x^2 - y^2 - 4y = 0$
11. $x^2 + 2x - y = 3$
12. $x + 2y + 2y^2 = 1$
13. $x^2 - 2y^2 + 3x + 4y = 2$
14. $9x^2 + 4y^2 - 18x + 8y = -13$
15. $9x^2 + 4y^2 - 18x + 8y = 23$
16. Identify and sketch the curve that is the graph of the equation $(x - y)^2 - (x + y)^2 = 1$.

*17. Light rays in the xy-plane coming from the point $(3, 4)$ reflect in a parabola so that they form a beam parallel to the x-axis. The parabola passes through the origin. Find its equation. (There are two possible answers.)

18. Light rays in the xy-plane coming from the origin are reflected by an ellipse so that they converge at the point $(3, 0)$. Find all possible equations for the ellipse.

In Exercises 19–22, identify the conic and find its centre, principal axes, foci, and eccentricity. Specify the asymptotes of any hyperbolas.

19. $xy + x - y = 2$

*20. $x^2 + 2xy + y^2 = 4x - 4y + 4$

*21. $8x^2 + 12xy + 17y^2 = 20$

*22. $x^2 - 4xy + 4y^2 + 2x + y = 0$

23. The *focus-directrix definition of a conic* defines a conic as a set of points P in the plane that satisfy the condition
$$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to } D} = \varepsilon,$$
where F is a fixed point, D a fixed straight line, and ε a fixed positive number. The conic is an ellipse, a parabola, or a hyperbola according to whether $\varepsilon < 1$, $\varepsilon = 1$, or $\varepsilon > 1$. Find the equation of the conic if F is the origin and D is the line $x = -p$.

Another parameter associated with conics is the **semi-latus rectum**, usually denoted ℓ. For a circle it is equal to the radius. For other conics it is half the length of the chord through a focus and perpendicular to the axis (for a parabola), the major axis (for an ellipse), or the transverse axis (for a hyperbola). That chord is called the **latus rectum** of the conic.

24. Show that the semi-latus rectum of the parabola is twice the distance from the vertex to the focus.
25. Show that the semi-latus rectum for an ellipse with semi-major axis a and semi-minor axis b is $\ell = b^2/a$.
26. Show that the formula in the above exercise also gives the semi-latus rectum of a hyperbola with semi-transverse axis a and semi-conjugate axis b.

*27. Suppose a plane intersects a right-circular cone in an ellipse and that two spheres (one on each side of the plane) are inscribed between the cone and the plane so that each is tangent to the cone around a circle and is also tangent to the plane at a point. Show that the points where these two spheres touch the plane are the foci of the ellipse. *Hint:* All tangent lines drawn to a sphere from a given point outside the sphere are equal in length. The distance between the two circles in which the spheres intersect the cone, measured along generators of the cone (i.e., straight lines lying on the cone), is the same for all generators.

*28. State and prove a result analogous to that in Exercise 27 but pertaining to a hyperbola.

*29. Suppose a plane intersects a right-circular cone in a parabola with vertex at V. Suppose that a sphere is inscribed between the cone and the plane as in the previous exercises and is tangent to the plane of the parabola at point F. Show that the chord to the parabola through F which is perpendicular to FV has length equal to that of the latus rectum of the parabola. Therefore, F is the focus of the parabola.

8.2 Parametric Curves

Suppose that an object moves around in the xy-plane so that the coordinates of its position at any time t are continuous functions of the variable t:
$$x = f(t), \qquad y = g(t).$$

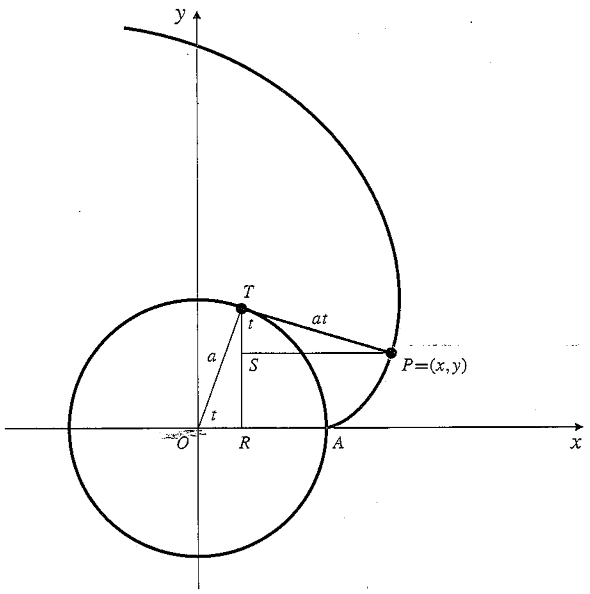

Figure 8.22 An involute of a circle

Exercises 8.2

In Exercises 1–10, sketch the given parametric curve, showing its direction with an arrow. Eliminate the parameter to give a Cartesian equation in x and y whose graph contains the parametric curve.

1. $x = 1 + 2t,\ y = t^2,\ (-\infty < t < \infty)$
2. $x = 2 - t,\ y = t + 1,\ (0 \le t < \infty)$
3. $x = \dfrac{1}{t},\ y = t - 1,\ (0 < t < 4)$
4. $x = \dfrac{1}{1+t^2},\ y = \dfrac{t}{1+t^2},\ (-\infty < t < \infty)$
5. $x = 3\sin 2t,\ y = 3\cos 2t,\ \left(0 \le t \le \dfrac{\pi}{3}\right)$
6. $x = a\sec t,\ y = b\tan t,\ \left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$
7. $x = 3\sin \pi t,\ y = 4\cos \pi t,\ (-1 \le t \le 1)$
8. $x = \cos\sin s,\ y = \sin\sin s,\ (-\infty < s < \infty)$
9. $x = \cos^3 t,\ y = \sin^3 t,\ (0 \le t \le 2\pi)$
10. $x = 1 - \sqrt{4 - t^2},\ y = 2 + t,\ (-2 \le t \le 2)$
11. Describe the parametric curve $x = \cosh t$, $y = \sinh t$, and find its Cartesian equation.
12. Describe the parametric curve $x = 2 - 3\cosh t$, $y = -1 + 2\sinh t$.
13. Describe the curve $x = t\,\cos t$, $y = t\,\sin t$, $(0 \le t \le 4\pi)$.
14. Show that each of the following sets of parametric equations represents a different arc of the parabola with equation $2(x + y) = 1 + (x - y)^2$.
    (a) $x = \cos^4 t,\ y = \sin^4 t$
    (b) $x = \sec^4 t,\ y = \tan^4 t$
    (c) $x = \tan^4 t,\ y = \sec^4 t$
15. Find a parametrization of the parabola $y = x^2$ using as parameter the slope of the tangent line at the general point.
16. Find a parametrization of the circle $x^2 + y^2 = R^2$ using as parameter the slope m of the line joining the general point to the point $(R, 0)$. Does the parametrization fail to give any point on the circle?
17. A circle of radius a is centred at the origin O. T is a point on the circle such that OT makes angle t with the positive x-axis. The tangent to the circle at T meets the x-axis at X. The point $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through T. Find, in terms of the parameter t, parametric equations for the curve $\mathcal{C}$ traced out by P as T moves around the circle. Also, eliminate t and find an equation for $\mathcal{C}$ in x and y. Sketch $\mathcal{C}$.
18. Repeat Exercise 17 with the following modification: OT meets a second circle of radius b centred at O at the point Y. $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through Y.

*19. **(The folium of Descartes)** Eliminate the parameter from the parametric equations

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3} \qquad (t \ne -1),$$

and hence find an ordinary equation in x and y for this curve. The parameter t can be interpreted as the slope of the line joining the general point (x, y) to the origin. Sketch the curve and show that the line $x + y = -1$ is an asymptote.

*20. **(A prolate cycloid)** A railroad wheel has a flange extending below the level of the track on which the wheel rolls. If the radius of the wheel is a and that of the flange is $b > a$, find parametric equations of the path of a point P at the circumference of the flange as the wheel rolls along the track. (Note that for a portion of each revolution of the wheel, P is moving backward.) Try to sketch the graph of this prolate cycloid.

*21. **(Hypocycloids)** If a circle of radius b rolls, without slipping, around the inside of a fixed circle of radius $a > b$, a point on the circumference of the rolling circle traces a curve called a hypocycloid. If the fixed circle is centred at the origin and the point tracing the curve starts at $(a, 0)$, show that the hypocycloid has parametric equations

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right),$$
$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right),$$

where t is the angle between the positive x-axis and the line from the origin to the point at which the rolling circle touches the fixed circle.

If $a = 2$ and $b = 1$, show that the hypocycloid becomes a straight line segment.

If $a = 4$ and $b = 1$, show that the parametric equations of the hypocycloid simplify to $x = 4\cos^3 t$, $y = 4\sin^3 t$. This curve is called a hypocycloid of four cusps or an **astroid**. (See Figure 8.23.) It has Cartesian equation $x^{2/3} + y^{2/3} = 4^{2/3}$.

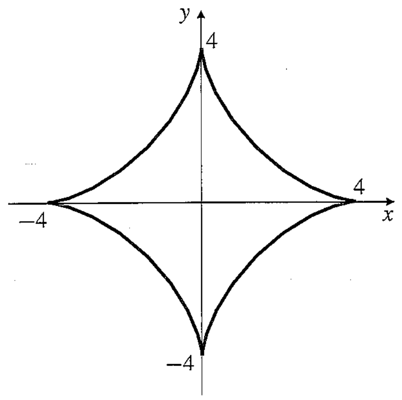

Figure 8.23 The astroid $x^{2/3} + y^{2/3} = 4^{2/3}$

Hypocycloids resemble the curves produced by a popular children's toy called Spirograph, but Spirograph curves result from following a point inside the disc of the rolling circle rather than on its circumference, and they therefore do not have sharp cusps.

*22. **(The witch of Agnesi)**

(a) Show that the curve traced out by the point P constructed from a circle as shown in Figure 8.24 has parametric equations $x = \tan t$, $y = \cos^2 t$ in terms of the angle t shown. (*Hint:* You will need to make extensive use of similar triangles.)

(b) Use a trigonometric identity to eliminate t from the parametric equations, and hence find an ordinary Cartesian equation for the curve.

This curve is named for the Italian mathematician Maria Agnesi (1718-1799), one of the foremost women scholars of her century and author of an important calculus text. The term *witch* is due to a mistranslation of the Italian word *versiera* ("turning curve"), which she used to describe the curve. The word is similar to *avversiera* ("wife of the devil" or "witch").

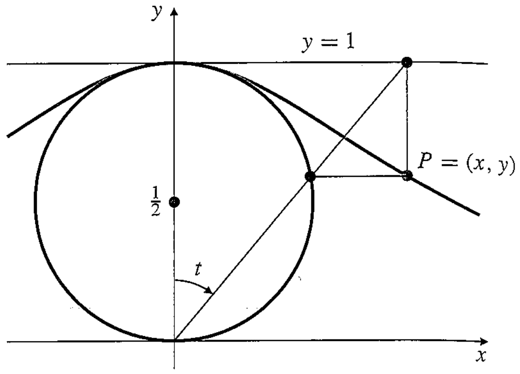

Figure 8.24 The witch of Agnesi

In Exercises 23–26, obtain a graph of the curve $x = \sin(mt)$, $y = \sin(nt)$ for the given values of m and n. Such curves are called **Lissajous figures.** They arise in the analysis of electrical signals using an oscilloscope. A signal of fixed but unknown frequency is applied to the vertical input, and a control signal is applied to the horizontal input. The horizontal frequency is varied until a stable Lissajous figure is observed. The (known) frequency of the control signal and the shape of the figure then determine the unknown frequency.

23. $m = 1, \quad n = 2$

24. $m = 1, \quad n = 3$

25. $m = 2, \quad n = 3$

26. $m = 2, \quad n = 5$

27. (Epicycloids) Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t - \frac{1}{n}\cos(nt)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin(nt)$$

for various integer and fractional values of $n \geq 3$. Can you formulate any principles governing the behaviour of such curves?

28. (More hypocycloids) Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t + \frac{1}{n}\cos((n-1)t)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin((n-1)t)$$

for various integer and fractional values of $n \geq 3$. Can you formulate any principles governing the behaviour of these curves?

8.3 Smooth Parametric Curves and Their Slopes

We say that a plane curve is *smooth* if it has a tangent line at each point P and this tangent turns in a continuous way as P moves along the curve. (That is, the angle is horizontal; at points where $dx/dt = 0$ but $dy/dt \neq 0$, the tangent is vertical. For points where $dx/dt = dy/dt = 0$, anything can happen; it is wise to calculate left- and right-hand limits of the slope dy/dx as the parameter t approaches one of these points. Concavity can be determined using the formula obtained above. We illustrate these ideas by reconsidering a parametric curve encountered in the previous section.

Example 4 Use slope and concavity information to sketch the graph of the parametric curve

$$x = f(t) = t^3 - 3t, \qquad y = g(t) = t^2, \qquad (-2 \le t \le 2)$$

previously encountered in Example 5 of Section 8.2.

Solution We have

$$f'(t) = 3(t^2 - 1) = 3(t-1)(t+1), \qquad g'(t) = 2t.$$

The curve has a horizontal tangent at $t = 0$, that is, at $(0, 0)$, and vertical tangents at $t = \pm 1$, that is, at $(2, 1)$ and $(-2, 1)$. Directional information for the curve between these points is summarized in the following chart.

t	-2		-1		0		1		2
$f'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$g'(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
x		$\rightarrow$	$\cdot$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\cdot$	$\rightarrow$	
y		$\downarrow$	$\downarrow$	$\downarrow$	$\cdot$	$\uparrow$	$\uparrow$	$\uparrow$	
curve		$\searrow$	$\downarrow$	$\swarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$	

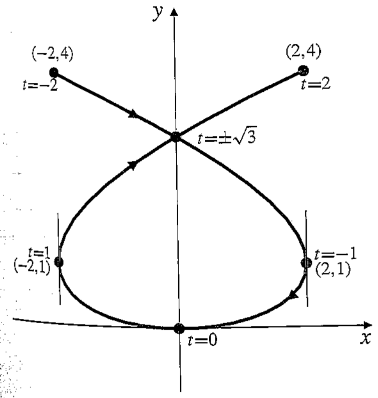

Figure 8.26

For concavity we calculate the second derivative d^2y/dx^2 by the formula obtained above. Since $f''(t) = 6t$ and $g''(t) = 2$, we have

$$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3} = \frac{3(t^2-1)(2) - 2t(6t)}{[3(t^2-1)]^3} = -\frac{2}{9}\frac{t^2+1}{(t^2-1)^3},$$

which is never zero but which fails to be defined at $t = \pm 1$. Evidently the curve is concave upward for $-1 < t < 1$ and concave downward elsewhere. The curve is sketched in Figure 8.26. ■

Exercises 8.3

In Exercises 1–8, find the coordinates of the points at which the given parametric curve has (a) a horizontal tangent and (b) a vertical tangent.

1. $x = t^2 + 1, \ y = 2t - 4$
2. $x = t^2 - 2t, \ y = t^2 + 2t$
3. $x = t^2 - 2t, \ y = t^3 - 12t$
4. $x = t^3 - 3t, \ y = 2t^3 + 3t^2$
5. $x = te^{-t^2/2}, \ y = e^{-t^2}$
6. $x = \sin t, \ y = \sin t - t\cos t$
7. $x = \sin 2t, \ y = \sin t$
8. $x = \dfrac{3t}{1+t^3}, \ y = \dfrac{3t^2}{1+t^3}$

Find the slopes of the curves in Exercises 9–12 at the points indicated.

9. $x = t^3 + t, \ y = 1 - t^3$, at $t = 1$
10. $x = t^4 - t^2, \ y = t^3 + 2t$, at $t = -1$
11. $x = \cos 2t, \ y = \sin t$, at $t = \pi/6$

12. $x = e^{2t}$, $y = te^{2t}$, at $t = -2$

Find parametric equations of the tangents to the curves in Exercises 13–14 at the indicated points.

13. $x = t^3 - 2t$, $y = t + t^3$, at $t = 1$

14. $x = t - \cos t$, $y = 1 - \sin t$, at $t = \pi/4$

15. Show that the curve $x = t^3 - t$, $y = t^2$ has two different tangent lines at the point $(0, 1)$ and find their slopes.

16. Find the slopes of two lines that are tangent to $x = \sin t$, $y = \sin 2t$ at the origin.

Where, if anywhere, do the curves in Exercises 17–20 fail to be smooth?

17. $x = t^3$, $y = t^2$

18. $x = (t-1)^4$, $y = (t-1)^3$

19. $x = t \sin t$, $y = t^3$

20. $x = t^3$, $y = t - \sin t$

In Exercises 21–25, sketch the graphs of the given parametric curves, making use of information from the first two derivatives. Unless otherwise stated, the parameter interval for each curve is the whole real line.

21. $x = t^2 - 2t$, $y = t^2 - 4t$

22. $x = t^3$, $y = 3t^2 - 1$

23. $x = t^3 - 3t$, $y = \dfrac{2}{1+t^2}$

24. $x = t^3 - 3t - 2$, $y = t^2 - t - 2$

25. $x = \cos t + t \sin t$, $y = \sin t - t \cos t$, $(t \ge 0)$. (See Example 9 of Section 8.2.)

8.4 Arc Lengths and Areas for Parametric Curves

In this section we look at the problems of finding lengths of curves defined parametrically, areas of surfaces of revolution obtained by rotating parametric curves, and areas of plane regions bounded by parametric curves.

Arc Lengths and Surface Areas

Let $\mathcal{C}$ be a smooth parametric curve with equations

$$x = f(t), \qquad y = g(t), \qquad (a \le t \le b).$$

(We assume that $f'(t)$ and $g'(t)$ are continuous on the interval $[a, b]$ and are never both zero.) From the differential triangle with legs dx and dy and hypotenuse ds (see Figure 8.27), we obtain $(ds)^2 = (dx)^2 + (dy)^2$, so we have

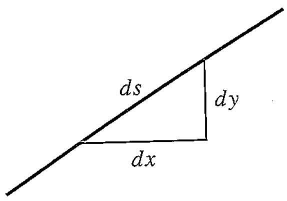

Figure 8.27 A differential triangle

The arc length element for a parametric curve

$$ds = \frac{ds}{dt}\,dt = \sqrt{\left(\frac{ds}{dt}\right)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The length of the curve $\mathcal{C}$ is given by

$$s = \int_{t=a}^{t=b} ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Example 1 Find the length of the parametric curve

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad (0 \le t \le 2).$$

Solution We have

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \qquad \frac{dy}{dt} = e^t(\sin t + \cos t).$$

Squaring these formulas, adding and simplifying, we get

$$\begin{aligned}\left(\frac{ds}{dt}\right)^2 &= e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2\\ &= e^{2t}\left(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t\right)\\ &= 2e^{2t}.\end{aligned}$$

Exercises 8.4

Find the lengths of the curves in Exercises 1–8.

1. $x = 3t^2,\ y = 2t^3,\ (0 \le t \le 1)$
2. $x = 1 + t^3,\ y = 1 - t^2,\ (-1 \le t \le 2)$
3. $x = a\cos^3 t,\ y = a\sin^3 t,\ (0 \le t \le 2\pi)$
4. $x = \ln(1 + t^2),\ y = 2\tan^{-1} t,\ (0 \le t \le 1)$
5. $x = t^2 \sin t,\ y = t^2 \cos t,\ (0 \le t \le 2\pi)$
6. $x = \cos t + t\sin t,\ y = \sin t - t\cos t,\ (0 \le t \le 2\pi)$
7. $x = t + \sin t,\ y = \cos t,\ (0 \le t \le \pi)$
8. $x = \sin^2 t,\ y = 2\cos t,\ (0 \le t \le \pi/2)$
9. Find the length of one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (One arch corresponds to $0 \le t \le 2\pi$.)
10. Find the area of the surfaces obtained by rotating one arch of the cycloid in Exercise 9 about (a) the x-axis, (b) the y-axis.
11. Find the area of the surface generated by rotating the curve $x = e^t \cos t$, $y = e^t \sin t$, $(0 \le t \le \pi/2)$ about the x-axis.
12. Find the area of the surface generated by rotating the curve of Exercise 11 about the y-axis.
13. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the y-axis.
14. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the x-axis.

In Exercises 15–20, sketch and find the area of the region R described in terms of the given parametric curves.

15. R is the closed loop bounded by $x = t^3 - 4t$, $y = t^2$, $(-2 \le t \le 2)$.
16. R is bounded by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$.
17. R is bounded by the coordinate axes and the parabolic arc $x = \sin^4 t$, $y = \cos^4 t$.
18. R is bounded by $x = \cos s \sin s$, $y = \sin^2 s$, $(0 \le s \le \pi/2)$, and the y-axis.
19. R is bounded by the oval $x = (2 + \sin t)\cos t$, $y = (2 + \sin t)\sin t$.

*20. R is bounded by the x-axis, the hyperbola $x = \sec t$, $y = \tan t$, and the ray joining the origin to the point $(\sec t_0, \tan t_0)$.

21. Show that the region bounded by the x-axis and the hyperbola $x = \cosh t$, $y = \sinh t$ (where $t > 0$), and the ray from the origin to the point $(\cosh t_0, \sinh t_0)$ has area $t_0/2$ square units. This proves a claim made at the beginning of Section 3.6.
22. Find the volume of the solid obtained by rotating about the x-axis the region bounded by that axis and one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (See Example 8 in Section 8.2.)
23. Find the volume generated by rotating about the x-axis the region lying under the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ and above the x-axis.

8.5 Polar Coordinates and Polar Curves

The **polar coordinate system** is an alternative to the rectangular (Cartesian) coordinate system for describing the location of points in a plane. Sometimes it is more important to know how far, and in what direction, a point is from the origin than it is to know its Cartesian coordinates. In the polar coordinate system there is an origin (or **pole**), O, and a **polar axis**, a ray (i.e., a half-line) extending from O horizontally to the right. The position of any point P in the plane is then determined by its polar coordinates $[r, \theta]$, where

(i) r is the distance from O to P, and

(ii) θ is the angle that the ray OP makes with the polar axis (counterclockwise angles being considered positive).

We will use square brackets for polar coordinates of a point to distinguish them from rectangular (Cartesian) coordinates. Figure 8.33 shows some points with their polar coordinates. The rectangular coordinate axes x and y are usually shown on a polar graph. The polar axis coincides with the positive x-axis.

Unlike rectangular coordinates, the polar coordinates of a point are not unique. The polar coordinates $[r, \theta_1]$ and $[r, \theta_2]$ represent the same point provided θ_1 and θ_2 differ by an integer multiple of 2π:

$$\theta_2 = \theta_1 + 2n\pi, \qquad \text{where } n = 0, \pm1, \pm2, \ldots$$

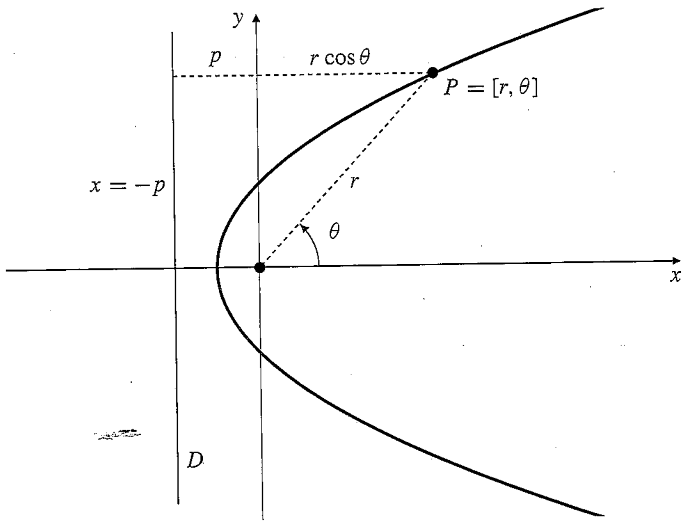

Figure 8.46 A conic curve with eccentricity ε, focus at the origin, and directrix $x = -p$

$$r = \frac{\varepsilon p}{1 - \varepsilon \cos\theta}.$$

Examples of the three possibilities (ellipse, parabola, and hyperbola) are shown in Figures 8.47–8.49. Note that for the hyperbola, the directions of the asymptotes are the angles that make the denominator $1 - \varepsilon \cos\theta = 0$. We will have more to say about polar equations of conics, especially ellipses, in Section 11.6.

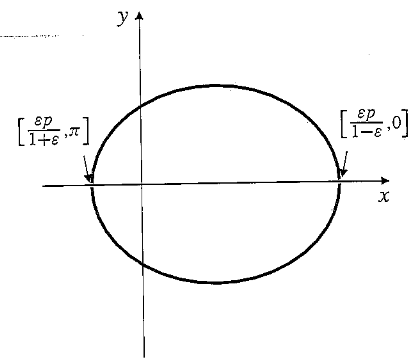

Figure 8.47 Ellipse: $\varepsilon < 1$

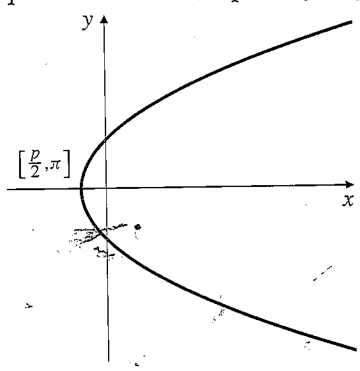

Figure 8.48 Parabola: $\varepsilon = 1$

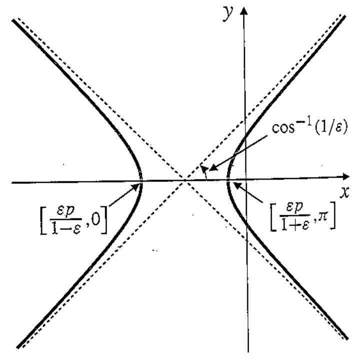

Figure 8.49 Hyperbola: $\varepsilon > 1$

Exercises 8.5

In Exercises 1–12, transform the given polar equation to rectangular coordinates, and identify the curve represented.

1. $r = 3\sec\theta$

2. $r = -2\csc\theta$

3. $r = \dfrac{5}{3\sin\theta - 4\cos\theta}$

4. $r = \sin\theta + \cos\theta$

5. $r^2 = \csc 2\theta$

6. $r = \sec\theta \tan\theta$

7. $r = \sec\theta(1 + \tan\theta)$

8. $r = \dfrac{2}{\sqrt{\cos^2\theta + 4\sin^2\theta}}$

9. $r = \dfrac{1}{1 - \cos\theta}$

10. $r = \dfrac{2}{2 - \cos\theta}$

11. $r = \dfrac{2}{1 - 2\sin\theta}$

12. $r = \dfrac{2}{1 + \sin\theta}$

In Exercises 13–24, sketch the polar graphs of the given equations.

13. $r = 1 + \sin\theta$

14. $r = 1 - \cos(\theta + \frac{\pi}{4})$

15. $r = 1 + 2\cos\theta$

16. $r = 1 - 2\sin\theta$

17. $r = 2 + \cos\theta$

18. $r = 2\sin 2\theta$

19. $r = \cos 3\theta$

20. $r = 2\cos 4\theta$

21. $r^2 = 4\sin 2\theta$

22. $r^2 = 4\cos 3\theta$

23. $r^2 = \sin 3\theta$

24. $r = \ln\theta$

Find all intersections of the pairs of curves in Exercises 25–28.

25. $r = \sqrt{3}\cos\theta, \quad r = \sin\theta$

26. $r^2 = 2\cos(2\theta), \quad r = 1$

27. $r = 1 + \cos\theta, \quad r = 3\cos\theta$

***28.** $r = \theta, \quad r = \theta + \pi$

29. Sketch the graph of the equation $r = 1/\theta$, $\theta > 0$. Show that this curve has a horizontal asymptote. Does $r = 1/(\theta - \alpha)$ have an asymptote?

30. How many leaves does the curve $r = \cos n\theta$ have? the curve $r^2 = \cos n\theta$? Distinguish the cases where n is odd and even.

31. Show that the polar graph $r = f(\theta)$ (where f is continuous) can be written as a parametric curve with parameter θ.

In Exercises 32–37, use computer graphing software or a graphing calculator to plot various members of the given families of polar curves, and try to observe patterns that would enable you to predict behaviour of other members of the families.

32. $r = \cos\theta\cos(m\theta)$, $m = 1, 2, 3, \ldots$

33. $r = 1 + \cos\theta\cos(m\theta)$, $m = 1, 2, 3, \ldots$

34. $r = \sin(2\theta)\sin(m\theta)$, $m = 2, 3, 4, 5, \ldots$

35. $r = 1 + \sin(2\theta)\sin(m\theta)$, $m = 2, 3, 4, 5, \ldots$

36. $r = C + \cos\theta\cos(2\theta)$ for $C = 0$, $C = 1$, values of C between 0 and 1, and values of C greater than 1

37. $r = C + \cos\theta\sin(3\theta)$ for $C = 0$, $C = 1$, values of C between 0 and 1, values of C less than 0, and values of C greater than 1

38. Plot the curve $r = \ln\theta$ for $0 < \theta \le 2\pi$. It intersects itself at point P. Thus there are two values θ_1 and θ_2 between 0 and 2π for which $[f(\theta_1), \theta_1] = [f(\theta_2), \theta_2]$. What equations must be satsified by θ_1 and θ_2? Find θ_1 and θ_2, and find the Cartesian coordinates of P correct to 6 decimal places.

39. Simultaneously plot the two curves $r = \ln\theta$ and $r = 1/\theta$, for $0 < \theta \le 2\pi$. The two curves intersect at two points. What equations must be satisfied by the θ values of these points? What are their Cartesian coordinates to 6 decimal places?

8.6 Slopes, Areas, and Arc Lengths for Polar Curves

There is a simple formula that can be used to determine the direction of the tangent line to a polar curve $r = f(\theta)$ at a point $P = [r, \theta]$ other than the origin. Let Q be a point on the curve near P corresponding to polar angle $\theta + h$. Let S be on OQ with PS perpendicular to OQ. Observe that $PS = f(\theta)\sin h$ and $SQ = OQ - OS = f(\theta + h) - f(\theta)\cos h$. If the tangent line to $r = f(\theta)$ at P makes angle ψ (Greek "psi") with the radial line OP as shown in Figure 8.50, then ψ is the limit of the angle SQP as $h \to 0$. Thus

$$\begin{aligned}
\tan\psi = \lim_{h\to 0}\frac{PS}{SQ} &= \lim_{h\to 0}\frac{f(\theta)\sin h}{f(\theta+h) - f(\theta)\cos h} \qquad \left[\frac{0}{0}\right]\\
&= \lim_{h\to 0}\frac{f(\theta)\cos h}{f'(\theta+h) + f(\theta)\sin h} \qquad \text{(by l'Hôpital's Rule)}\\
&= \frac{f(\theta)}{f'(\theta)} = \frac{r}{dr/d\theta}.
\end{aligned}$$

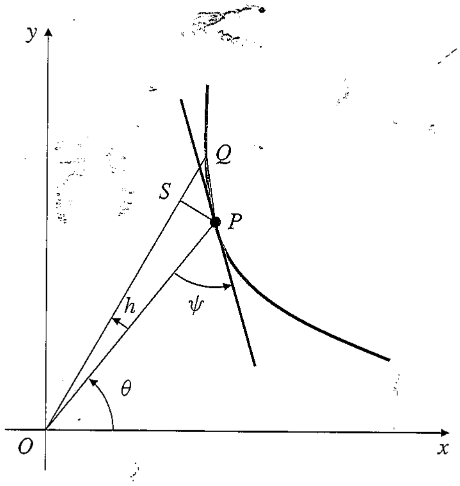

Figure 8.50 The angle ψ is the limit of angle SQP as $h \to 0$

Tangent direction for a polar curve

At any point P other than the origin on the polar curve $r = f(\theta)$, the angle ψ between the radial line from the origin to P and the tangent to the curve is given by

$$\tan\psi = \frac{f(\theta)}{f'(\theta)}.$$

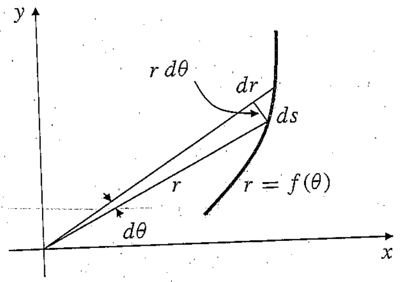

Figure 8.55 The arc length element for a polar curve

so we obtain the following formula:

Arc length element for a polar curve

The arc length element for the polar curve $r = f(\theta)$ is

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\,d\theta.$$

This arc length element can also be derived from that for a parametric curve. See Exercise 26 at the end of this section.

Example 4 Find the total length of the cardioid $r = a(1 + \cos\theta)$.

Solution The total length is twice the length from $\theta = 0$ to $\theta = \pi$. (Review Figure 8.53.) Since $dr/d\theta = -a\sin\theta$ for the cardioid, the arc length is

$$\begin{aligned} s &= 2\int_0^{\pi} \sqrt{a^2\sin^2\theta + a^2(1+\cos\theta)^2}\,d\theta \\ &= 2\int_0^{\pi} \sqrt{2a^2 + 2a^2\cos\theta}\,d\theta \qquad \text{(but } 1+\cos\theta = 2\cos^2(\theta/2)\text{)} \\ &= 2\sqrt{2}a\int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}}\,d\theta \\ &= 4a\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta = 8a\sin\frac{\theta}{2}\bigg|_0^{\pi} = 8a \text{ units.} \end{aligned}$$

Exercises 8.6

In Exercises 1–11, sketch and find the areas of the given polar regions R.

1. R lies between the origin and the spiral $r = \sqrt{\theta}$, $0 \le \theta \le 2\pi$.
2. R lies between the origin and the spiral $r = \theta$, $0 \le \theta \le 2\pi$.
3. R is bounded by the curve $r^2 = a^2\cos 2\theta$.
4. R is one leaf of the curve $r = \sin 3\theta$.
5. R is bounded by the curve $r = \cos 4\theta$.
6. R lies inside both of the circles $r = a$ and $r = 2a\cos\theta$.
7. R lies inside the cardioid $r = 1 - \cos\theta$ and outside the circle $r = 1$.
8. R lies inside the cardioid $r = a(1 - \sin\theta)$ and inside the circle $r = a$.
9. R lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.
10. R is bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$.
11. R is bounded by the smaller loop of the curve $r = 1 + 2\cos\theta$.

Find the lengths of the polar curves in Exercises 12–14.

12. $r = \theta^2$, $0 \le \theta \le \pi$
13. $r = e^{a\theta}$, $-\pi \le \theta \le \pi$
14. $r = a\theta$, $0 \le \theta \le 2\pi$
15. Show that the total arc length of the lemniscate $r^2 = \cos 2\theta$ is $4\int_0^{\pi/4} \sqrt{\sec 2\theta}\,d\theta$.
16. One leaf of the lemniscate $r^2 = \cos 2\theta$ is rotated (a) about the x-axis and (b) about the y-axis. Find the area of the surface generated in each case.

*17. Determine the angles at which the straight line $\theta = \pi/4$ intersects the cardioid $r = 1 + \sin\theta$.

*18. At what points do the curves $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$ intersect? At what angle do the curves intersect at each of these points?

*19. At what points do the curves $r = 1 - \cos\theta$ and $r = 1 - \sin\theta$ intersect? At what angle do the curves intersect at each of these points?

In Exercises 20–25, find all points on the given curve where the tangent line is horizontal, vertical, or does not exist.

*20. $r = \cos\theta + \sin\theta$

*21. $r = 2\cos\theta$

*22. $r^2 = \cos 2\theta$

*23. $r = \sin 2\theta$

*24. $r = e^{\theta}$

*25. $r = 2(1 - \sin\theta)$

26. The polar curve $r = f(\theta)$, $(\alpha \le \theta \le \beta)$, can be parametrized:

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Derive the formula for the arc length element for the polar curve from that for a parametric curve.

Chapter Review

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ a conic section
  - ◇ an ellipse
  - ◇ a parabola
  - ◇ a hyperbola
  - ◇ a parametric curve
  - ◇ a parametrization of a curve
  - ◇ a smooth curve
  - ◇ a polar curve
- **What is the focus-directrix definition of a conic?**
- **How can you find the slope of a parametric curve?**
- **How can you find the length of a parametric curve?**
- **How can you find the length of a polar curve?**
- **How can you find the area bounded by a polar curve?**

Review Exercises

In Exercises 1–4, describe the conic having the given equation. Give its foci and principal axes and, if it is a hyperbola, its asymptotes.

1. $x^2 + 2y^2 = 2$
2. $9x^2 - 4y^2 = 36$
3. $x + y^2 = 2y + 3$
4. $2x^2 + 8y^2 = 4x - 48y$

Identify the parametric curves in Exercises 5–10.

5. $x = t,\ y = 2 - t,\ (0 \le t \le 2)$
6. $x = 2\sin 3t,\ y = 2\cos 3t,\ (0 \le t \le 1/2)$
7. $x = \cosh t,\ y = \sinh^2 t,$
8. $x = e^t,\ y = e^{-2t},\ (-1 \le t \le 1)$
9. $x = \cos(t/2),\ y = 4\sin(t/2),\ (0 \le t \le \pi)$
10. $x = \cos t + \sin t,\ y = \cos t - \sin t,\ (0 \le t \le 2\pi)$

In Exercises 11–14, determine the points where the given parametric curves have horizontal and vertical tangents, and sketch the curves.

11. $x = \dfrac{4}{1+t^2},\ y = t^3 - 3t$
12. $x = t^3 - 3t,\ y = t^3 + 3t$
13. $x = t^3 - 3t,\ y = t^3$
14. $x = t^3 - 3t,\ y = t^3 - 12t$
15. Find the area bounded by the part of the curve $x = t^3 - t$, $y = |t^3|$ that forms a closed loop.
16. Find the volume of the solid generated by rotating the closed loop in Exercise 15 about the y-axis.
17. Find the length of the curve $x = e^t - t$, $y = 4e^{t/2}$ from $t = 0$ to $t = 2$.
18. Find the area of the surface obtained by rotating the arc in Exercise 17 about the x-axis.

Sketch the polar graphs of the equations in Exercises 19–24.

19. $r = \theta,\ \left(-\frac{3\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$
20. $r = |\theta|,\ (-2\pi \le \theta \le 2\pi)$
21. $r = 1 + \cos 2\theta$
22. $r = 2 + \cos 2\theta$
23. $r = 1 + 2\cos 2\theta$
24. $r = 1 - \sin 3\theta$
25. Find the area of one of the two larger loops of the curve in Exercise 23.
26. Find the area of one of the two smaller loops of the curve in Exercise 23.
27. Find the area of the smaller of the two loops enclosed by the curve $r = 1 + \sqrt{2}\sin\theta$.
28. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and to the left of the line $x = 1/4$.

Challenging Problems

1. A glass in the shape of a circular cylinder of radius 4 cm is more than half filled with water. If the glass is tilted by an angle θ from the vertical, where θ is small enough that no water spills out, find the surface area of the water.
2. Show that a plane that is not parallel to the axis of a circular cylinder intersects the cylinder in an ellipse. *Hint:* You can do this by the same method used in Exercise 27 of Section 8.1.
3. Given two points F_1 and F_2 that are foci of an ellipse and a third point P on the ellipse, describe a geometric method (using a straight edge and a compass) for constructing the tangent line to the ellipse at P. *Hint:* Think about the reflection property of ellipses.
4. Let C be a parabola with vertex V, and let P be any point on the parabola. Let R be the point where the tangent to the parabola at P intersects the axis of the parabola. (Thus the axis is the line RV.) Let Q be the point on RV such that PQ is perpendicular to RV. Show that V bisects the line segment RQ. How does this result suggest a geometric method for constructing a tangent to a parabola at a point on it, given the axis and vertex of the parabola?
5. A barrel has the shape of a solid of revolution obtained by rotating about its major axis the part of an ellipse lying between lines through its foci perpendicular to that axis. The barrel is 4 ft high and 2 ft in radius at its middle. What is its volume?
6. (a) Show that any straight line not passing through the origin can be written in polar form as
$$r = \frac{a}{\cos(\theta - \theta_0)},$$
where a and θ_0 are constants. What is the geometric significance of these constants?

   (b) Let $r = g(\theta)$ be the polar equation of a straight line that does not pass through the origin. Show that
$$g^2 + 2(g')^2 - gg'' = 0.$$

   (c) Let $r = f(\theta)$ be the polar equation of a curve, where f'' is continuous and $r \ne 0$ in some interval of values of θ. Let
$$F = f^2 + 2(f')^2 - ff''.$$
Show that the curve is turning toward the origin if $F > 0$ and away from the origin if $F < 0$. *Hint:* Let $r = g(\theta)$ be the polar equation of a straight line tangent to the curve, and use part (b). How do f, f', and f'' relate to g, g', and g'' at the point of tangency?

7. (Fast trip, but it might get hot) If we assume that the density of the earth is uniform throughout, then it can be shown that the acceleration of gravity at a distance $r \le R$ from the centre of the earth is directed toward the centre of the earth and has magnitude $a(r) = rg/R$, where g is the usual acceleration of gravity at the surface ($g \approx 32$ ft/s^2), and R is the radius of the earth ($R \approx 3960$ mi). Suppose that a straight tunnel AB is drilled through the earth between any two points A and B on the surface, say Atlanta and Baghdad. (See Figure 8.56.)

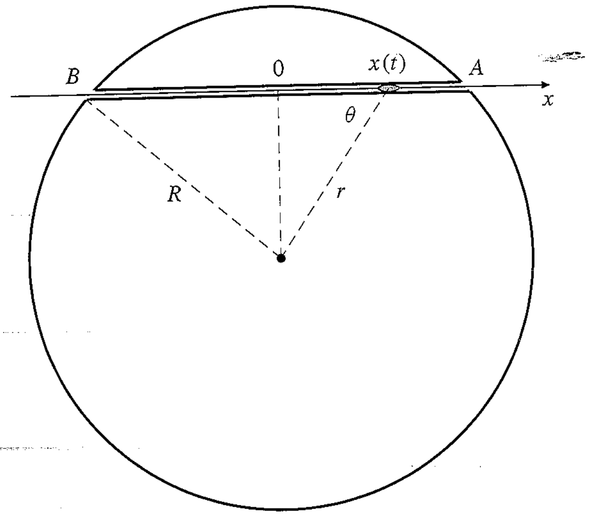

Figure 8.56

Suppose that a vehicle is constructed that can slide without friction or air resistance through this tunnel. Show that such a vehicle will, if released at one end of the tunnel, fall back and forth between A and B, executing simple harmonic motion with period $2\pi\sqrt{R/g}$. How many minutes will the round trip take? What is surprising here is that this period does not depend on where A and B are or on the distance between them. *Hint:* Let the x-axis lie along the tunnel, with origin at the point closest to the centre of the earth. When the vehicle is at position with x-coordinate $x(t)$, its acceleration along the tunnel is the component of the gravitational acceleration along the tunnel, that is, $-a(r)\cos\theta$, where θ is the angle between the line of the tunnel and the line from the vehicle to the centre of the earth.

* **8. (Search and Rescue)** Two coast guard stations pick up a distress signal from a ship and use radio direction finders to locate it. Station O observes that the distress signal is coming from the northeast (45° east of north), while station P, which is 100 miles north of station O, observes that the signal is coming from due east. Each station's direction finder is accurate to within $\pm 3°$.

(a) How large an area of the ocean must a rescue aircraft search to ensure that it finds the foundering ship?

(b) If the accuracy of the direction finders is within $\pm\varepsilon$, how sensitive is the search area to changes in ε when $\varepsilon = 3°$? (Express your answer in square miles per degree.)

9. Figure 8.57 shows the graphs of the parametric curve $x = \sin t$, $y = \frac{1}{2}\sin(2t)$, $0 \le t \le 2\pi$, and the polar curve $r^2 = \cos(2\theta)$. Each has the shape of an "∞." Which curve is which? Find the area inside the outer curve and outside the inner curve.

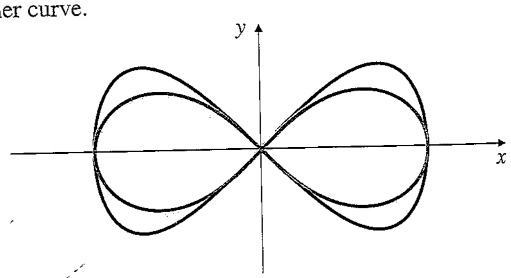

Figure 8.57