EXERCISES 4.3

Evaluate the limits in Exercises 1–32.

1. $\lim_{x\to 0} \frac{3x}{\tan 4x}$

2. $\lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4}$

3. $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

4. $\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

5. $\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

6. $\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$

7. $\lim_{x\to 0} x \cot x$

8. $\lim_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)}$

9. $\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi}$

10. $\lim_{x\to 0} \frac{10^x - e^x}{x}$

11. $\lim_{x\to \pi/2} \frac{\cos 3x}{\pi - 2x}$

12. $\lim_{x\to 1} \frac{\ln(ex)-1}{\sin \pi x}$

13. $\lim_{x\to \infty} x \sin \frac{1}{x}$

14. $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

15. $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

16. $\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$

17. $\lim_{x\to 0+} \frac{\sin^2 x}{\tan x - x}$

18. $\lim_{r\to \pi/2} \frac{\ln \sin r}{\cos r}$

19. $\lim_{t\to \pi/2} \frac{\sin t}{t}$

20. $\lim_{x\to 1-} \frac{\arccos x}{x-1}$

21. $\lim_{x\to \infty} x(2\tan^{-1} x - \pi)$

22. $\lim_{t\to (\pi/2)-} (\sec t - \tan t)$

23. $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{te^{at}}\right)$

24. $\lim_{x\to 0+} x^{\sqrt{x}}$

25. $\lim_{x\to 0+} (\csc x)^{\sin^2 x}$

26. $\lim_{x\to 1+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

27. $\lim_{t\to 0} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t}$

28. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

29. $\lim_{t\to 0} (\cos 2t)^{1/t^2}$

30. $\lim_{x\to 0+} \frac{\csc x}{\ln x}$

31. $\lim_{x\to 1-} \frac{\ln \sin \pi x}{\csc \pi x}$

32. $\lim_{x\to 0} (1+\tan x)^{1/x}$

33. **(A Newton quotient for the second derivative)** Evaluate $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ if f is a twice differentiable function.

34. If f has a continuous third derivative, evaluate

$$\lim_{h\to 0} \frac{f(x+3h)-3f(x+h)+3f(x-h)-f(x-3h)}{h^3}.$$

35. **(Proof of the second l'Hôpital Rule)** Fill in the details of the following outline of a proof of the second l'Hôpital Rule (Theorem 4) for the case where a and L are both finite. Let $a < x < t < b$ and show that there exists c in (x, t) such that

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)}.$$

Now juggle the above equation algebraically into the form

$$\frac{f(x)}{g(x)} - L = \frac{f'(c)}{g'(c)} - L + \frac{1}{g(x)}\left(f(t) - g(t)\frac{f'(c)}{g'(c)}\right).$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f'(c)}{g'(c)} - L\right| + \frac{1}{|g(x)|}\left(|f(t)| + |g(t)|\left|\frac{f'(c)}{g'(c)}\right|\right).$$

Now show that the right side of the above inequality can be made as small as you wish (say less than a positive number ϵ) by choosing first t and then x close enough to a. Remember, you are given that $\lim_{c\to a+} \left(f'(c)/g'(c)\right) = L$ and $\lim_{x\to a+} |g(x)| = \infty$.

4.4 Extreme Values

The first derivative of a function is a source of much useful information about the behaviour of the function. As we have already seen, the sign of f' tells us whether f is increasing or decreasing. In this section we use this information to find maximum and minimum values of functions. In Section 4.8 we will put the techniques developed here to use solving problems that require finding maximum and minimum values.

Maximum and Minimum Values

Recall (from Section 1.4) that a function has a maximum value at x_0 if $f(x) \le f(x_0)$ for all x in the domain of f. The maximum value is $f(x_0)$. To be more precise, we should call such a maximum value an *absolute* or *global* maximum because it is the largest value that f attains anywhere on its entire domain.

Solution We have

$$\lim_{x\to 0+} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

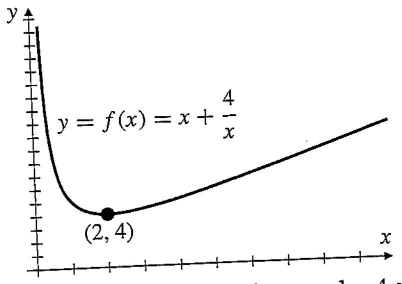

Figure 4.24 f has minimum value 4 at $x = 2$

Since $f(1) = 5 < \infty$, Theorem 8 guarantees that f must have an absolute minimum value at some point in $(0, \infty)$. To find the minimum value we must check the values of f at any critical points or singular points in the interval. We have

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2},$$

which equals 0 only at $x = 2$ and $x = -2$. Since f has domain $(0, \infty)$, it has no singular points and only one critical point, namely, $x = 2$, where f has the value $f(2) = 4$. This must be the minimum value of f on $(0, \infty)$. (See Figure 4.24.)

EXAMPLE 6 Let $f(x) = x e^{-x^2}$. Find and classify the critical points of f, evaluate $\lim_{x\to\pm\infty} f(x)$, and use these results to help you sketch the graph of f.

Solution $f'(x) = e^{-x^2}(1 - 2x^2) = 0$ only if $1 - 2x^2 = 0$ since the exponential is always positive. Thus, the critical points are $\pm\frac{1}{\sqrt{2}}$. We have $f\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2e}}$. f' is positive (or negative) when $1 - 2x^2$ is positive (or negative). We summarize the intervals where f is increasing and decreasing in chart form:

		CP		CP	
x		$-1/\sqrt{2}$		$1/\sqrt{2}$	→
f'	−	0	+	0	−
f	↘	min	↗	max	↘

Note that $f(0) = 0$ and that f is an odd function ($f(-x) = -f(x)$), so the graph is symmetric about the origin. Also,

$$\lim_{x\to\pm\infty} x e^{-x^2} = \left(\lim_{x\to\pm\infty} \frac{1}{x}\right)\left(\lim_{x\to\pm\infty} \frac{x^2}{e^{x^2}}\right) = 0 \times 0 = 0$$

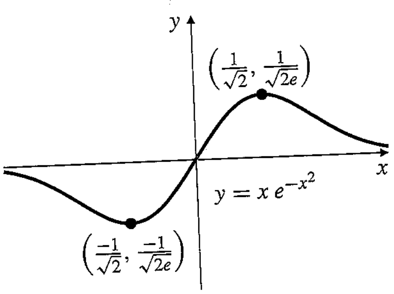

Figure 4.25 The graph for Example 6

because $\lim_{x\to\pm\infty} x^2 e^{-x^2} = \lim_{u\to\infty} u e^{-u} = 0$ by Theorem 5 of Section 3.4. Since $f(x)$ is positive at $x = 1/\sqrt{2}$ and is negative at $x = -1/\sqrt{2}$, f must have absolute maximum and minimum values by Theorem 8. These values can only be the values $\pm 1/\sqrt{2e}$ at the two critical points. The graph is shown in Figure 4.25. The x-axis is an asymptote as $x \to \pm\infty$.

EXERCISES 4.4

In Exercises 1–17, determine whether the given function has any local or absolute extreme values, and find those values if possible.

1. $f(x) = x + 2$ on $[-1, 1]$
2. $f(x) = x + 2$ on $(-\infty, 0]$
3. $f(x) = x + 2$ on $[-1, 1)$
4. $f(x) = x^2 - 1$
5. $f(x) = x^2 - 1$ on $[-2, 3]$
6. $f(x) = x^2 - 1$ on $(2, 3)$
7. $f(x) = x^3 + x - 4$ on $[a, b]$
8. $f(x) = x^3 + x - 4$ on (a, b)
9. $f(x) = x^5 + x^3 + 2x$ on $(a, b]$
10. $f(x) = \dfrac{1}{x-1}$
11. $f(x) = \dfrac{1}{x-1}$ on $(0, 1)$
12. $f(x) = \dfrac{1}{x-1}$ on $[2, 3]$
13. $f(x) = |x - 1|$ on $[-2, 2]$

14. $|x^2 - x - 2|$ on $[-3, 3]$

15. $f(x) = \dfrac{1}{x^2+1}$

16. $f(x) = (x+2)^{2/3}$

17. $f(x) = (x-2)^{1/3}$

In Exercises 18–40, locate and classify all local extreme values of the given function. Determine whether any of these extreme values are absolute. Sketch the graph of the function.

18. $f(x) = x^2 + 2x$

19. $f(x) = x^3 - 3x - 2$

20. $f(x) = (x^2 - 4)^2$

21. $f(x) = x^3(x-1)^2$

22. $f(x) = x^2(x-1)^2$

23. $f(x) = x(x^2-1)^2$

24. $f(x) = \dfrac{x}{x^2+1}$

25. $f(x) = \dfrac{x^2}{x^2+1}$

26. $f(x) = \dfrac{x}{\sqrt{x^4+1}}$

27. $f(x) = x\sqrt{2-x^2}$

28. $f(x) = x + \sin x$

29. $f(x) = x - 2\sin x$

30. $f(x) = x - 2\tan^{-1} x$

31. $f(x) = 2x - \sin^{-1} x$

32. $f(x) = e^{-x^2/2}$

33. $f(x) = x\,2^{-x}$

34. $f(x) = x^2 e^{-x^2}$

35. $f(x) = \dfrac{\ln x}{x}$

36. $f(x) = |x+1|$

37. $f(x) = |x^2 - 1|$

38. $f(x) = \sin|x|$

39. $f(x) = |\sin x|$

40. $f(x) = (x-1)^{2/3} - (x+1)^{2/3}$

In Exercises 41–46, determine whether the given function has absolute maximum or absolute minimum values. Justify your answers. Find the extreme values if you can.

41. $\dfrac{x}{\sqrt{x^2+1}}$

42. $\dfrac{x}{\sqrt{x^4+1}}$

43. $x\sqrt{4-x^2}$

44. $\dfrac{x^2}{\sqrt{4-x^2}}$

45. $\dfrac{1}{x\sin x}$ on $(0, \pi)$

46. $\dfrac{\sin x}{x}$

47. If a function has an absolute maximum value, must it have any local maximum values? If a function has a local maximum value, must it have an absolute maximum value? Give reasons for your answers.

48. If the function f has an absolute maximum value and $g(x) = |f(x)|$, must g have an absolute maximum value? Justify your answer.

49. **(A function with no max or min at an endpoint)** Let

$$f(x) = \begin{cases} x\sin\dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ but that it has neither a local maximum nor a local minimum value at the endpoint $x = 0$.

4.5 Concavity and Inflections

Like the first derivative, the second derivative of a function also provides useful information about the behaviour of the function and the shape of its graph: it determines whether the graph is *bending upward* (i.e., has increasing slope) or *bending downward* (i.e., has decreasing slope) as we move along the graph toward the right.

DEFINITION 3

We say that the function f is **concave up** on an open interval I if it is differentiable there and the derivative f' is an increasing function on I. Similarly, f is **concave down** on I if f' exists and is decreasing on I.

The terms "concave up" and "concave down" are used to describe the graph of the function as well as the function itself.

Note that concavity is defined only for differentiable functions, and even for those, only on intervals on which their derivatives are not constant. According to the above definition, a function is neither concave up nor concave down on an interval where its graph is a straight line segment. We say the function has no concavity on such an interval. We also say a function has opposite concavity on two intervals if it is concave up on one interval and concave down on the other.

The function f whose graph is shown in Figure 4.26 is concave up on the interval (a, b) and concave down on the interval (b, c).

Some geometric observations can be made about concavity:

(i) If f is concave up on an interval, then, on that interval, the graph of f lies above its tangents, and chords joining points on the graph lie above the graph.

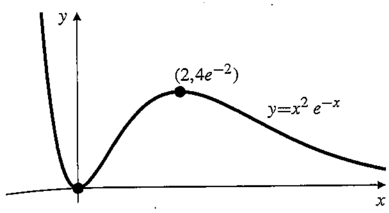

Figure 4.33 The critical points of $f(x) = x^2 e^{-x}$

Solution We begin by calculating the first two derivatives of f:

$$f'(x) = (2x - x^2)e^{-x} = x(2-x)e^{-x} = 0 \quad \text{at } x = 0 \text{ and } x = 2,$$
$$f''(x) = (2 - 4x + x^2)e^{-x}$$
$$f''(0) = 2 > 0, \qquad f''(2) = -2e^{-2} < 0.$$

Thus, f has a local minimum value at $x = 0$ and a local maximum value at $x = 2$. See Figure 4.33.

For many functions the second derivative is more complicated to calculate than the first derivative, so the First Derivative Test is likely to be of more use in classifying critical points than is the Second Derivative Test. Also note that the First Derivative Test can classify local extreme values that occur at endpoints and singular points as well as at critical points.

It is possible to generalize the Second Derivative Test to obtain a higher derivative test to deal with some situations where the second derivative is zero at a critical point. (See Exercise 40 at the end of this section.)

EXERCISES 4.5

In Exercises 1–22, determine the intervals of constant concavity of the given function, and locate any inflection points.

1. $f(x) = \sqrt{x}$
2. $f(x) = 2x - x^2$
3. $f(x) = x^2 + 2x + 3$
4. $f(x) = x - x^3$
5. $f(x) = 10x^3 - 3x^5$
6. $f(x) = 10x^3 + 3x^5$
7. $f(x) = (3 - x^2)^2$
8. $f(x) = (2 + 2x - x^2)^2$
9. $f(x) = (x^2 - 4)^3$
10. $f(x) = \dfrac{x}{x^2 + 3}$
11. $f(x) = \sin x$
12. $f(x) = \cos 3x$
13. $f(x) = x + \sin 2x$
14. $f(x) = x - 2\sin x$
15. $f(x) = \tan^{-1} x$
16. $f(x) = x\,e^x$
17. $f(x) = e^{-x^2}$
18. $f(x) = \dfrac{\ln(x^2)}{x}$
19. $f(x) = \ln(1 + x^2)$
20. $f(x) = (\ln x)^2$
21. $f(x) = \dfrac{x^3}{3} - 4x^2 + 12x - \dfrac{25}{3}$
22. $f(x) = (x-1)^{1/3} + (x+1)^{1/3}$
23. Discuss the concavity of the linear function $f(x) = ax + b$. Does it have any inflections?

Classify the critical points of the functions in Exercises 24–35 using the Second Derivative Test whenever possible.

24. $f(x) = 3x^3 - 36x - 3$
25. $f(x) = x(x-2)^2 + 1$
26. $f(x) = x + \dfrac{4}{x}$
27. $f(x) = x^3 + \dfrac{1}{x}$
28. $f(x) = \dfrac{x}{2^x}$
29. $f(x) = \dfrac{x}{1 + x^2}$
30. $f(x) = xe^x$
31. $f(x) = x\ln x$
32. $f(x) = (x^2 - 4)^2$
33. $f(x) = (x^2 - 4)^3$
34. $f(x) = (x^2 - 3)e^x$
35. $f(x) = x^2 e^{-2x^2}$
36. Let $f(x) = x^2$ if $x \ge 0$ and $f(x) = -x^2$ if $x < 0$. Is 0 a critical point of f? Does f have an inflection point there? Is $f''(0) = 0$? If a function has a nonvertical tangent line at an inflection point, does the second derivative of the function necessarily vanish at that point?
37. Verify that if f is concave up on an interval, then its graph lies above its tangent lines on that interval. *Hint:* Suppose f is concave up on an open interval containing x_0. Let $h(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$. Show that h has a local minimum value at x_0 and hence that $h(x) \ge 0$ on the interval. Show that $h(x) > 0$ if $x \ne x_0$.
38. Verify that the graph $y = f(x)$ crosses its tangent line at an inflection point. *Hint:* Consider separately the cases where the tangent line is vertical and nonvertical.
39. Let $f_n(x) = x^n$ and $g_n(x) = -x^n$, $(n = 2, 3, 4, \ldots)$. Determine whether each function has a local maximum, a local minimum, or an inflection point at $x = 0$.
40. **(Higher Derivative Test)** Use your conclusions from Exercise 39 to suggest a generalization of the Second Derivative Test that applies when
$$f'(x_0) = f''(x_0) = \ldots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \ne 0,$$
for some $k \ge 2$.
41. This problem shows that no test based solely on the signs of derivatives at x_0 can determine whether every function with a critical point at x_0 has a local maximum or minimum or an inflection point there. Let
$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Prove the following:
(a) $\lim_{x\to 0} x^{-n} f(x) = 0$ for $n = 0, 1, 2, 3, \ldots$.
(b) $\lim_{x\to 0} P(1/x) f(x) = 0$ for every polynomial P.

(c) For $x \neq 0$, $f^{(k)}(x) = P_k(1/x) f(x) (k = 1, 2, 3, \ldots)$, where P_k is a polynomial.

(d) $f^{(k)}(0)$ exists and equals 0 for $k = 1, 2, 3, \ldots$.

(e) f has a local minimum at $x = 0$; $-f$ has a local maximum at $x = 0$.

(f) If $g(x) = x f(x)$, then $g^{(k)}(0) = 0$ for every positive integer k and g has an inflection point at $x = 0$.

42. A function may have neither a local maximum nor a local minimum nor an inflection at a critical point. Show this by considering the following function:

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(0) = f(0) = 0$, so the x-axis is tangent to the graph of f at $x = 0$; but $f'(x)$ is not continuous at $x = 0$, so $f''(0)$ does not exist. Show that the concavity of f is not constant on any interval with endpoint 0.

4.6 Sketching the Graph of a Function

When sketching the graph $y = f(x)$ of a function f, we have three sources of useful information:

(i) **the function f itself**, from which we determine the coordinates of some points on the graph, the symmetry of the graph, and any asymptotes;

(ii) **the first derivative, f'**, from which we determine the intervals of increase and decrease and the location of any local extreme values; and

(iii) **the second derivative, f''**, from which we determine the concavity and inflection points, and sometimes extreme values.

Items (ii) and (iii) were explored in the previous two sections. In this section we consider what we can learn from the function itself about the shape of its graph, and then we illustrate the entire sketching procedure with several examples using all three sources of information.

We could sketch a graph by plotting the coordinates of many points on it and joining them by a suitably smooth curve. This is what computer software and graphics calculators do. When carried out by hand (without a computer or calculator), this simplistic approach is at best tedious and at worst can fail to reveal the most interesting aspects of the graph (singular points, extreme values, and so on). We could also compute the slope at each of the plotted points and, by drawing short line segments through these points with the appropriate slopes, ensure that the sketched graph passes through each plotted point with the correct slope. A more efficient procedure is to obtain the coordinates of only a few points and use qualitative information from the function and its first and second derivatives to determine the *shape* of the graph between these points.

Besides critical and singular points and inflections, a graph may have other "interesting" points. The **intercepts** (points at which the graph intersects the coordinate axes) are usually among these. When sketching any graph it is wise to try to find all such intercepts, that is, all points with coordinates $(x, 0)$ and $(0, y)$ that lie on the graph. Of course, not every graph will have such points, and even when they do exist it may not always be possible to compute them exactly. Whenever a graph is made up of several disconnected pieces (called **components**), the coordinates of *at least one point on each component* must be obtained. It can sometimes be useful to determine the slopes at those points too. Vertical asymptotes (discussed below) usually break the graph of a function into components.

Realizing that a given function possesses some symmetry can aid greatly in obtaining a good sketch of its graph. In Section P.4 we discussed odd and even functions and observed that odd functions have graphs that are symmetric about the origin, while even functions have graphs that are symmetric about the y-axis, as shown in Figure 4.34. These are the symmetries you are most likely to notice, but functions can have other symmetries. For example, the graph of $2 + (x - 1)^2$ will certainly be symmetric about

EXAMPLE 9 Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$. (See Figure 4.42.)

Solution $f'(x) = \dfrac{4}{3}\dfrac{x}{(x^2-1)^{1/3}}, \quad f''(x) = \dfrac{4}{9}\dfrac{x^2-3}{(x^2-1)^{4/3}}.$

From f: Domain: all x.
Asymptotes: none. ($f(x)$ grows like $x^{4/3}$ as $x \to \pm\infty$.)
Symmetry: about the y-axis (f is an even function).
Intercepts: $(\pm 1, 0)$, $(0, 1)$.

From f': Critical points: $x = 0$; singular points: $x = \pm 1$.

From f'': $f''(x) = 0$ at $x = \pm\sqrt{3}$; points $(\pm\sqrt{3}, 2^{2/3}) \approx (\pm 1.73, 1.59)$;
$f''(x)$ not defined at $x = \pm 1$.

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	$\searrow$		$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$		$\nearrow$
	$\smile$	infl	$\frown$		$\frown$		$\frown$		$\frown$	infl	$\smile$

EXERCISES 4.6

1. Figure 4.43 shows the graphs of a function f, its two derivatives f' and f'', and another function g. Which graph corresponds to each function?

2. List, for each function graphed in Figure 4.43, such information that you can determine (approximately) by inspecting the graph (e.g., symmetry, asymptotes, intercepts, intervals of increase and decrease, critical and singular points, local maxima and minima, intervals of constant concavity, inflection points).

3. Figure 4.44 shows the graphs of four functions:

$$f(x) = \frac{x}{1-x^2}, \qquad g(x) = \frac{x^3}{1-x^4},$$
$$h(x) = \frac{x^3 - x}{\sqrt{x^6+1}}, \qquad k(x) = \frac{x^3}{\sqrt{|x^4-1|}}.$$

Which graph corresponds to each function?

4. Repeat Exercise 2 for the graphs in Figure 4.44.

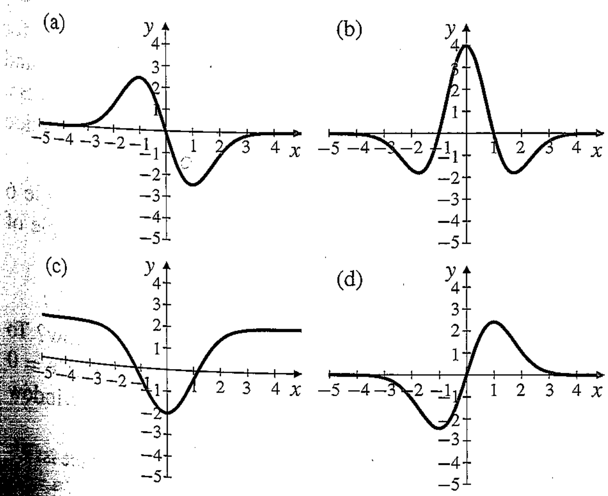

Figure 4.43

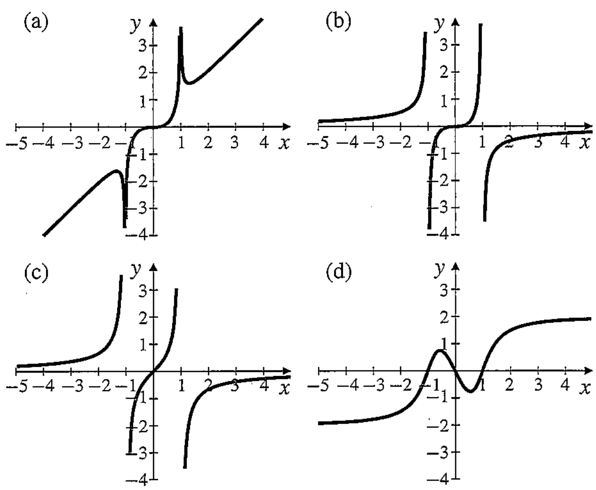

Figure 4.44

In Exercises 5–6, sketch the graph of a function that has the given properties. Identify any critical points, singular points, local maxima and minima, and inflection points. Assume that f is continuous and its derivatives exist everywhere unless the contrary is implied or explicitly stated.

5. $f(0) = 1$, $f(\pm 1) = 0$, $f(2) = 1$, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -1$, $f'(x) > 0$ on $(-\infty, 0)$ and on $(1, \infty)$, $f'(x) < 0$ on $(0, 1)$, $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, 2)$, and $f''(x) < 0$ on $(2, \infty)$.

6. $f(-1) = 0$, $f(0) = 2$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, $\lim_{x\to\pm\infty}(f(x) + 1 - x) = 0$, $f'(x) > 0$ on $(-\infty, -1)$, $(-1, 0)$ and $(2, \infty)$, $f'(x) < 0$ on $(0, 2)$, $\lim_{x\to-1} f'(x) = \infty$, $f''(x) > 0$ on $(-\infty, -1)$ and on $(1, 3)$, and $f''(x) < 0$ on $(-1, 1)$ and on $(3, \infty)$.

In Exercises 7–39, sketch the graphs of the given functions, making use of any suitable information you can obtain from the function and its first and second derivatives.

7. $y = (x^2 - 1)^3$

8. $y = x(x^2 - 1)^2$

9. $y = \dfrac{2 - x}{x}$

10. $y = \dfrac{x - 1}{x + 1}$

11. $y = \dfrac{x^3}{1 + x}$

12. $y = \dfrac{1}{4 + x^2}$

13. $y = \dfrac{1}{2 - x^2}$

14. $y = \dfrac{x}{x^2 - 1}$

15. $y = \dfrac{x^2}{x^2 - 1}$

16. $y = \dfrac{x^3}{x^2 - 1}$

17. $y = \dfrac{x^3}{x^2 + 1}$

18. $y = \dfrac{x^2}{x^2 + 1}$

19. $y = \dfrac{x^2 - 4}{x + 1}$

20. $y = \dfrac{x^2 - 2}{x^2 - 1}$

21. $y = \dfrac{x^3 - 4x}{x^2 - 1}$

22. $y = \dfrac{x^2 - 1}{x^2}$

23. $y = \dfrac{x^5}{(x^2 - 1)^2}$

24. $y = \dfrac{(2 - x)^2}{x^3}$

25. $y = \dfrac{1}{x^3 - 4x}$

26. $y = \dfrac{x}{x^2 + x - 2}$

27. $y = \dfrac{x^3 - 3x^2 + 1}{x^3}$

28. $y = x + \sin x$

29. $y = x + 2\sin x$

30. $y = e^{-x^2}$

31. $y = xe^x$

32. $y = e^{-x}\sin x$, $(x \ge 0)$

33. $y = x^2 e^{-x^2}$

34. $y = x^2 e^x$

35. $y = \dfrac{\ln x}{x}$, $(x > 0)$

36. $y = \dfrac{\ln x}{x^2}$, $(x > 0)$

37. $y = \dfrac{1}{\sqrt{4 - x^2}}$

38. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

39. $y = (x^2 - 1)^{1/3}$

40. What is $\lim_{x\to 0+} x \ln x$? $\lim_{x\to 0} x \ln |x|$? If $f(x) = x \ln |x|$ for $x \ne 0$, is it possible to define $f(0)$ in such a way that f is continuous on the whole real line? Sketch the graph of f.

41. What straight line is an asymptote of the curve $y = \dfrac{\sin x}{1 + x^2}$? At what points does the curve cross this asymptote?

4.7 Graphing with Computers

The techniques for sketching, developed in the previous section, are useful for graphs of functions that are simple enough to allow you to calculate and analyze their derivatives. They are also essential for testing the validity of graphs produced by computers or calculators, which can be inaccurate or misleading for a variety of reasons, including the case of numerical monsters introduced in previous chapters. In practice, it is often easiest to first produce a graph using a computer or graphing calculator, but many times this will not turn out to be the last step. (We will use the term "computer" for both computers and calculators.) For many simple functions this can be a quick and painless activity, but sometimes functions have properties that complicate the process. Knowledge of the function, from techniques like those above, is important to guide you on what the next steps must be.

The Maple command[1] for viewing the graph of the function from Example 6 of Section 4.6, together with its oblique asymptote, is a straightforward example of plotting; we ask Maple to plot both $(x^2 + 2x + 4)/(2x)$ and $1 + (x/2)$.

```
> plot({(x^2+2*x+4)/(2*x), 1+(x/2)}, x=-6..6, y=-7..7);
```

This command sets the window $-6 \le x \le 6$ and $-7 \le y \le 7$. Why that window? To get a plot that characterizes the function, knowledge of its vertical asymptote at $x = 0$ is essential. (If $x - 10$ were substituted for x in the expression, the given window

[1] Although we focus on Maple to illustrate the issues of graphing with computers, the issues presented are general ones, pertaining to all software and computers.

Remark **Modifying Example 4** Given the sparse information provided in the statement of the problem in Example 4, interpretations (i) and (ii) are the best we can do. The problem could be made more meaningful economically (from the point of view, say, of a tin can manufacturer) if more elements were brought into it. For example:

(a) Most cans use thicker material for the cylindrical wall than for the top and bottom disks. If the cylindrical wall material costs $\$A$ per unit area and the material for the top and bottom costs $\$B$ per unit area, we might prefer to minimize the total cost of materials for a can of given volume. What is the optimal shape if $A = 2B$?

(b) Large numbers of cans are to be manufactured. The material is probably being cut out of sheets of metal. The cylindrical walls are made by bending up rectangles, and rectangles can be cut from the sheet with little or no waste. There will, however, always be a proportion of material wasted when the disks are cut out. The exact proportion will depend on how the disks are arranged; two possible arrangements are shown in Figure 4.54. What is the optimal shape of the can if a square packing of disks is used? A hexagonal packing? Any such modification of the original problem will alter the optimal shape to some extent. In "real-world" problems, many factors may have to be taken into account to come up with a "best" strategy.

(c) The problem makes no provision for costs of manufacturing the can other than the cost of sheet metal. There may also be costs for joining the opposite edges of the rectangle to make the cylinder and for joining the top and bottom disks to the cylinder. These costs may be proportional to the lengths of the joins.

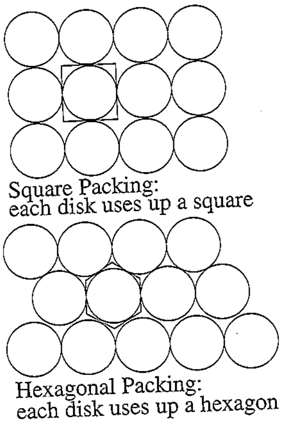

Figure 4.54 Square and hexagonal packing of disks in a plane

In most of the examples above, the maximum or minimum value being sought occurred at a critical point. Our final example is one where this is not the case.

EXAMPLE 5 A man can run twice as fast as he can swim. He is standing at point A on the edge of a circular swimming pool 40 m in diameter, and he wishes to get to the diametrically opposite point B as quickly as possible. He can run around the edge to point C, then swim directly from C to B. Where should C be chosen to minimize the total time taken to get from A to B?

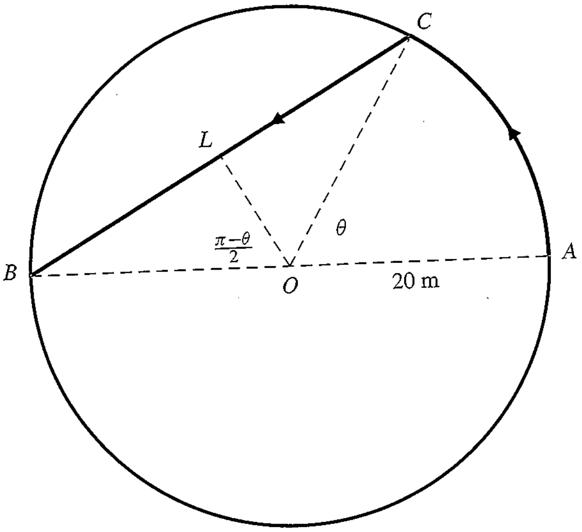

Figure 4.55 Running and swimming to get from A to B

Solution It is convenient to describe the position of C in terms of the angle AOC where O is the centre of the pool. (See Figure 4.55.) Let θ denote this angle. Clear $0 \le \theta \le \pi$. (If $\theta = 0$, the man swims the whole way; if $\theta = \pi$, he runs the who way.) The radius of the pool is 20 m, so arc $AC = 20\theta$. Since angle $BOC = \pi -$ we have angle $BOL = (\pi - \theta)/2$ and chord $BC = 2BL = 40\sin((\pi - \theta)/2)$.

Suppose the man swims at a rate k m/s and therefore runs at a rate $2k$ m/s. If t

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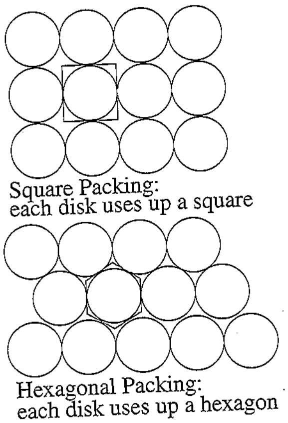

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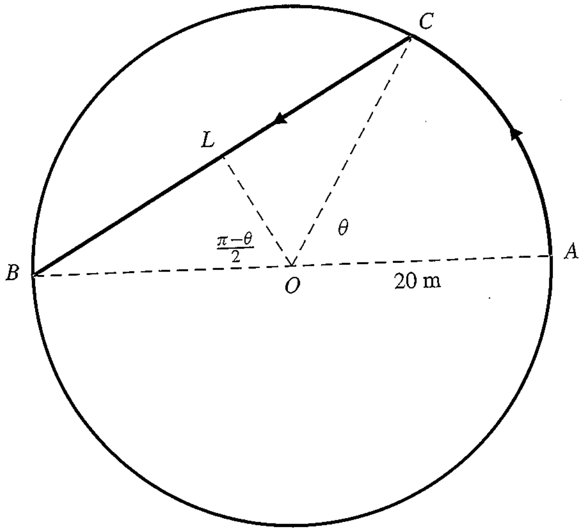

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Suppose the man swims at a rate k m/s and therefore runs at a rate $2k$ m/s.

Solution For $f(t) = t^{1/2}$, we have

$$f'(t) = \frac{1}{2}t^{-1/2} \qquad \text{and} \qquad f''(t) = -\frac{1}{4}t^{-3/2}.$$

For $25 < t < 26$, we have $f''(t) < 0$, so $\sqrt{26} = f(26) < L(26) = 5.1$. Also, $t^{3/2} > 25^{3/2} = 125$, so $|f''(t)| < (1/4)(1/125) = 1/500$ and

$$|E(26)| < \frac{1}{2} \times \frac{1}{500} \times (26-25)^2 = \frac{1}{1,000} = 0.001.$$

Therefore, $f(26) > L(26) - 0.001 = 5.099$, and $\sqrt{26}$ is in the interval $(5.099, 5.1)$.

Remark We can use Corollary C of Theorem 11 and the fact that $\sqrt{26} < 5.1$ to find a better (i.e., smaller) interval containing $\sqrt{26}$ as follows. If $25 < t < 26$, then $125 = 25^{3/2} < t^{3/2} < 26^{3/2} < 5.1^3$. Thus

$$M = -\frac{1}{4 \times 125} < f''(t) < -\frac{1}{4 \times 5.1^3} = N$$

$$\sqrt{26} \approx L(26) + \frac{M+N}{4} = 5.1 - \frac{1}{4}\left(\frac{1}{4 \times 125} + \frac{1}{4 \times 5.1^3}\right) \approx 5.099\,028\,8$$

$$|\text{Error}| < \frac{N-M}{4} = \frac{1}{16}\left(-\frac{1}{5.1^3} + \frac{1}{125}\right) \approx 0.000\,028\,8.$$

Thus $\sqrt{26}$ lies in the interval $(5.099\,00, 5.099\,06)$.

EXAMPLE 5 Use a suitable linearization to find an approximate value for $\cos 36° = \cos(\pi/5)$. Is the true value greater than or less than your approximation? Estimate the size of the error, and give an interval that you can be sure contains $\cos(36°)$.

Solution Let $f(t) = \cos t$, so that $f'(t) = -\sin t$ and $f''(t) = -\cos t$. The value of a nearest to 36° for which we know $\cos a$ is $a = 30° = \pi/6$, so we use the linearization about that point:

$$L(x) = \cos\frac{\pi}{6} - \sin\frac{\pi}{6}\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right).$$

Since $(\pi/5) - (\pi/6) = \pi/30$, our approximation is

$$\cos 36° = \cos\frac{\pi}{5} \approx L\left(\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{30}\right) \approx 0.813\,67.$$

If $(\pi/6) < t < (\pi/5)$, then $f''(t) < 0$ and $|f''(t)| < \cos(\pi/6) = \sqrt{3}/2$. Therefore, $\cos 36° < 0.813\,67$ and

$$|E(36°)| < \frac{\sqrt{3}}{4}\left(\frac{\pi}{30}\right)^2 < 0.004\,75.$$

Thus, $0.813\,67 - 0.004\,75 < \cos 36° < 0.813\,67$, so $\cos 36°$ lies in the interval $(0.808\,92, 0.813\,67)$.

Remark The error in the linearization of $f(x)$ about $x = a$ can be interpreted in terms of differentials (see Section 2.7 and the beginning of this section) as follows. If $\Delta x = dx = x - a$, then the change in $f(x)$ as we pass from $x = a$ to $x = a + \Delta x$ is $f(a + \Delta x) - f(a) = \Delta y$, and the corresponding change in the linearization $L(x)$ is $f'(a)(x-a) = f'(a)\,dx$, which is just the value at $x = a$ of the differential $dy = f'(x)\,dx$. Thus

$$E(x) = \Delta y - dy.$$

Solution Write the Taylor formula for e^x at $x = 0$ (from Table 5) with n replaced by $2n + 1$, and then rewrite that with x replaced by $-x$. We get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} + O\big(x^{2n+2}\big),$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} - \frac{x^{2n+1}}{(2n+1)!} + O\big(x^{2n+2}\big)$$

as $x \to 0$. Now average these two to get

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + O\big(x^{2n+2}\big)$$

as $x \to 0$. By Theorem 13 the Maclaurin polynomial $P_{2n}(x)$ for $\cosh x$ is

$$P_{2n}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!}.$$

EXAMPLE 7 Obtain the Taylor polynomial of order 3 for e^{2x} about $x = 1$ from the corresponding Maclaurin polynomial for e^x (from Table 5).

Solution Writing $x = 1 + (x - 1)$, we have

$$\begin{aligned} e^{2x} &= e^{2+2(x-1)} = e^2 e^{2(x-1)} \\ &= e^2\left[1 + 2(x-1) + \frac{2^2(x-1)^2}{2!} + \frac{2^3(x-1)^3}{3!} + O\big((x-1)^4\big)\right] \end{aligned}$$

as $x \to 1$. By Theorem 13 the Taylor polynomial $P_3(x)$ for e^{2x} at $x = 1$ must be

$$P_3(x) = e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2}{3}(x-1)^3.$$

EXAMPLE 8 Use the Taylor formula for $\ln(1 + x)$ (from Table 5) to find th Taylor polynomial $P_3(x)$ for $\ln x$ about $x = e$. (This provides a alternative to using the definition of Taylor polynomial as was done to solve the sam problem in Example 1(b).

Solution We have $x = e + (x - e) = e(1 + t)$ where $t = (x - e)/e$. As $x \to e$ w have $t \to 0$, so

$$\begin{aligned} \ln x &= \ln e + \ln(1+t) = \ln e + t - \frac{t^2}{2} + \frac{t^3}{3} + O(t^4) \\ &= 1 + \frac{x-e}{e} - \frac{1}{2}\left(\frac{x-e}{e}\right)^2 + \frac{1}{3}\left(\frac{x-e}{e}\right)^3 + O\big((x-e)^4\big). \end{aligned}$$

Therefore, by Theorem 13,

$$P_3(x) = 1 + \frac{x-e}{e} - \frac{1}{2}\left(\frac{x-e}{e}\right)^2 + \frac{1}{3}\left(\frac{x-e}{e}\right)^3.$$

Evaluating Limits of Indeterminate Forms

Taylor and Maclaurin polynomials provide us with another method for evaluati limits of indeterminate forms of type [0/0]. For some such limits this method can considerably easier than using l'Hôpital's Rule.

In Exercises 15–20, write the indicated case of Taylor's formula for the given function. What is the Lagrange remainder in each case?

15. $f(x) = \sin x,\ a = 0,\ n = 7$

16. $f(x) = \cos x,\ a = 0,\ n = 6$

17. $f(x) = \sin x,\ a = \pi/4,\ n = 4$

18. $f(x) = \dfrac{1}{1-x},\ a = 0,\ n = 6$

19. $f(x) = \ln x,\ a = 1, n = 6$

20. $f(x) = \tan x,\ a = 0,\ n = 3$

Find the requested Taylor polynomials in Exercises 21–26 by using known Taylor or Maclaurin polynomials and changing variables as in Examples 6–8.

21. $P_3(x)$ for e^{3x} about $x = -1$.

22. $P_8(x)$ for e^{-x^2} about $x = 0$.

23. $P_4(x)$ for $\sin^2 x$ about $x = 0$. *Hint:* $\sin^2 x = \dfrac{1-\cos(2x)}{2}$.

24. $P_5(x)$ for $\sin x$ about $x = \pi$.

25. $P_6(x)$ for $1/(1+2x^2)$ about $x = 0$

26. $P_8(x)$ for $\cos(3x - \pi)$ about $x = 0$.

27. Find all Maclaurin polynomials $P_n(x)$ for $f(x) = x^3$.

28. Find all Taylor polynomials $P_n(x)$ for $f(x) = x^3$ at $x = 1$.

29. Find the Maclaurin polynomial $P_{2n+1}(x)$ for $\sinh x$ by suitably combining polynomials for e^x and e^{-x}.

30. By suitably combining Maclaurin polynomials for $\ln(1+x)$ and $\ln(1-x)$, find the Maclaurin polynomial of order $2n+1$ for $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

31. Write Taylor's formula for $f(x) = e^{-x}$ with $a = 0$, and use it to calculate $1/e$ to 5 decimal places. (You may use a calculator but not the e^x function on it.)

32. Write the general form of Taylor's formula for $f(x) = \sin x$ at $x = 0$ with Lagrange remainder. How large need n be taken to ensure that the corresponding Taylor polynomial approximation will give the sine of 1 radian correct to 5 decimal places?

33. What is the best order 2 approximation to the function $f(x) = (x-1)^2$ at $x = 0$? What is the error in this approximation? Now answer the same questions for $g(x) = x^3 + 2x^2 + 3x + 4$. Can the constant $1/6 = 1/3!$, in the error formula for the degree 2 approximation, be improved (i.e., made smaller)?

34. By factoring $1 - x^{n+1}$ (or by long division), show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}. \qquad (*)$$

Next, show that if $|x| \le K < 1$, then

$$\left|\frac{x^{n+1}}{1-x}\right| \le \frac{1}{1-K}|x^{n+1}|.$$

This implies that $x^{n+1}/(1-x) = O(x^{n+1})$ as $x \to 0$ and confirms formula (d) of Table 5. What does Theorem 13 the say about the nth-order Maclaurin polynomial for $1/(1-x)$

35. By differentiating identity (*) in Exercise 34 and then replacing n with $n+1$, show that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \frac{n+2-(n+1)x}{(1-x)^2}x^{n+1}.$$

Then use Theorem 13 to determine the nth-order Maclaurin polynomial for $1/(1-x)^2$.

4.11 Roundoff Error, Truncation Error, and Computers

In Section 4.7 we introduced the idea of **roundoff error**, while in Sections 4.9 and 4.1 we discussed the result of approximating a function by its Taylor polynomials. T resulting error here is known as **truncation error.** This conventional terminology m be a bit confusing at first because rounding off is itself a kind of truncation of the digi representation of a number. However in numerical analysis "truncation" is reserved f discarding higher order terms, typically represented by big-O, often leaving a Tayl polynomial.

Truncation error is a crucial source of error in using computers to do mathem ical operations. In computation with computers, many of the mathematical functio and structures being investigated are approximated by polynomials in order to ma it possible for computers to manipulate them. However, the other source of err roundoff, is ubiquitous, so it is inevitable that mathematics on computers has to invol consideration of both sources of error. These sources can sometimes be treated in pendently, but in other circumstances they can interact with each other in fascinati ways. In this section we look at some of these fascinating interactions in the form Numerical Monsters using Maple. Of course, as stated previously, the issues conc all calculation on computers and not Maple in particular.

CHAPTER REVIEW

Key Ideas

- **What do the following words, phrases, and statements mean?**
  - ◇ critical point of f ◇ singular point of f
  - ◇ inflection point of f
  - ◇ f has absolute maximum value M
  - ◇ f has a local minimum value at $x = c$
  - ◇ vertical asymptote ◇ horizontal asymptote
  - ◇ oblique asymptote ◇ machine epsilon
  - ◇ the linearization of $f(x)$ about $x = a$
  - ◇ the Taylor polynomial of degree n of $f(x)$ about $x = a$
  - ◇ Taylor's formula with Lagrange remainder
  - ◇ $f(x) = O\big((x-a)^n\big)$ as $x \to a$
  - ◇ a root of $f(x) = 0$ ◇ a fixed point of $f(x)$
  - ◇ an indeterminate form ◇ l'Hôpital's Rules
- **Describe how to estimate the error in a linear (tangent line) approximation to the value of a function.**
- **Describe how to find a root of an equation $f(x) = 0$ by using Newton's Method. When will this method work well?**

Review Exercises

1. If the radius r of a ball is increasing at a rate of 2 percent per minute, how fast is the volume V of the ball increasing?

2. **(Gravitational attraction)** The gravitational attraction of the earth on a mass m at distance r from the centre of the earth is a continuous function of r for $r \geq 0$, given by

$$F = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \geq R \\ mkr & \text{if } 0 \leq r < R, \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

(a) Find the constant k in terms of g and R.

(b) F decreases as m moves away from the surface of the earth, either upward or downward. Show that F decreases as r increases from R at twice the rate at which F decreases as r decreases from R.

3. **(Resistors in parallel)** Two variable resistors R_1 and R_2 are connected in parallel so that their combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

At an instant when $R_1 = 250$ ohms and $R_2 = 1000$ ohms, R_1 is increasing at a rate of 100 ohms/min. How fast must R_2 be changing at that moment (a) to keep R constant? and (b) to enable R to increase at a rate of 10 ohms/min?

4. **(Gas law)** The volume V (in m^3), pressure P (in kilopascals, kPa), and temperature T (in kelvin, K) for a sample of a certain gas satisfy the equation $pV = 5.0T$.

(a) How rapidly does the pressure increase if the temperature is 400 K and increasing at 4 K/min while the gas is kept confined in a volume of 2.0 m^3?

(b) How rapidly does the pressure decrease if the volume is 2 m^3 and increases at 0.05 m^3/min while the temperature is kept constant at 400 K?

5. **(The size of a print run)** It costs a publisher \$10,000 to set up the presses for a print run of a book and \$8 to cover the material costs for each book printed. In addition, machinery servicing, labour, and warehousing add another $\$6.25 \times 10^{-7}x^2$ to the cost of each book if x copies are manufactured during the printing. How many copies should the publisher print in order to minimize the average cost per book?

6. **(Maximizing profit)** A bicycle wholesaler must pay the manufacturer \$75 for each bicycle. Market research tells the wholesaler that if she charges her customers \$$x$ per bicycle, she can expect to sell $N(x) = 4.5 \times 10^6/x^2$ of them. What price should she charge to maximize her profit, and how many bicycles should she order from the manufacturer?

7. Find the largest possible volume of a right-circular cone that can be inscribed in a sphere of radius R.

8. **(Minimizing production costs)** The cost \$$C(x)$ of production in a factory varies with the amount x of product manufactured. The cost may rise sharply with x for x small, and more slowly for larger values of x because of economies of scale. However, if x becomes too large, the resources of the factory can be overtaxed, and the cost can begin to rise quickly again. Figure 4.70 shows the graph of a typical such cost function $C(x)$.

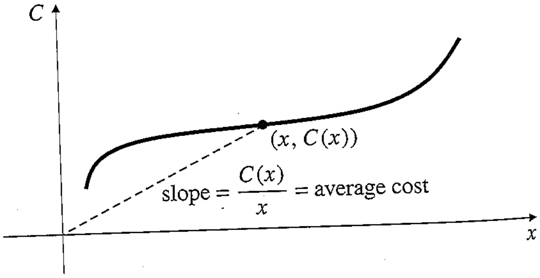

Figure 4.70

If x units are manufactured, the average cost per unit i[s] \$$C(x)/x$, which is the slope of the line from the origin t[o] the point $(x, C(x))$ on the graph.

(a) If it is desired to choose x to minimize this average co[st] per unit (as would be the case if all units produced cou[ld] be sold for the same price), show that x should be chose[n] to make the average cost equal to the marginal cost:

$$\frac{C(x)}{x} = C'(x).$$

(b) Interpret the conclusion of (a) geometrically in the figu[re].

(c) If the average cost equals the marginal cost for some [x,] does x necessarily minimize the average cost?

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  - ◇ the linearization of $f(x)$ about $x = a$
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  - ◇ Taylor's formula with Lagrange remainder
  - ◇ $f(x) = O\left((x-a)^n\right)$ as $x \to a$
  - ◇ a root of $f(x) = 0$
  - ◇ a fixed point of $f(x)$
  - ◇ an indeterminate form
  - ◇ l'Hôpital's Rules
- **Describe how to estimate the error in a linear (tangent line) approximation to the value of a function.**
- **Describe how to find a root of an equation $f(x) = 0$ by using Newton's Method. When will this method work well?**

Review Exercises

1. If the radius r of a ball is increasing at a rate of 2 percent per minute, how fast is the volume V of the ball increasing?

2. **(Gravitational attraction)** The gravitational attraction of the earth on a mass m at distance r from the centre of the earth is a continuous function of r for $r \geq 0$, given by

$$F = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \geq R \\ mkr & \text{if } 0 \leq r < R, \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

(a) Find the constant k in terms of g and R.

(b) F decreases as m moves away from the surface of the earth, either upward or downward. Show that F decreases as r increases from R at twice the rate at which F decreases as r decreases from R.

3. **(Resistors in parallel)** Two variable resistors R_1 and R_2 are connected in parallel so that their combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

At an instant when $R_1 = 250$ ohms and $R_2 = 1000$ ohms, R_1 is increasing at a rate of 100 ohms/min. How fast must R_2 be changing at that moment (a) to keep R constant? and (b) to enable R to increase at a rate of 10 ohms/min?

4. **(Gas law)** The volume V (in m^3), pressure P (in kilopascals, kPa), and temperature T (in kelvin, K) for a sample of a certain gas satisfy the equation $pV = 5.0T$.

(a) How rapidly does the pressure increase if the temperature is 400 K and increasing at 4 K/min while the gas is kept confined in a volume of 2.0 m^3?

(b) How rapidly does the pressure decrease if the volume is 2 m^3 and increases at 0.05 m^3/min while the temperature is kept constant at 400 K?

5. **(The size of a print run)** It costs a publisher \$10,000 to set up the presses for a print run of a book and \$8 to cover the material costs for each book printed. In addition, machinery servicing, labour, and warehousing add another $\$6.25 \times 10^{-7}x^2$ to the cost of each book if x copies are manufactured during the printing. How many copies should the publisher print in order to minimize the average cost per book?

6. **(Maximizing profit)** A bicycle wholesaler must pay the manufacturer \$75 for each bicycle. Market research tells the wholesaler that if she charges her customers \$$x$ per bicycle, she can expect to sell $N(x) = 4.5 \times 10^6/x^2$ of them. What price should she charge to maximize her profit, and how many bicycles should she order from the manufacturer?

7. Find the largest possible volume of a right-circular cone that can be inscribed in a sphere of radius R.

8. **(Minimizing production costs)** The cost \$$C(x)$ of production in a factory varies with the amount x of product manufactured. The cost may rise sharply with x for x small, and more slowly for larger values of x because of economies of scale. However, if x becomes too large, the resources of the factory can be overtaxed, and the cost can begin to rise quickly again. Figure 4.70 shows the graph of a typical such cost function $C(x)$.

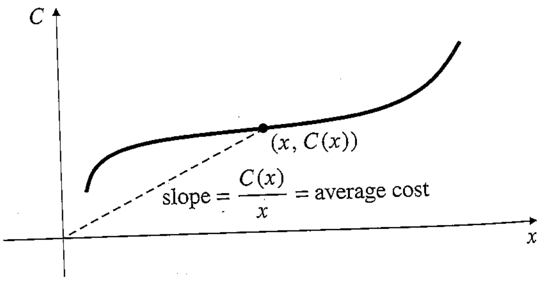

Figure 4.70

If x units are manufactured, the average cost per unit i \$$C(x)/x$, which is the slope of the line from the origin t the point $(x, C(x))$ on the graph.

(a) If it is desired to choose x to minimize this average co per unit (as would be the case if all units produced coul be sold for the same price), show that x should be chose to make the average cost equal to the marginal cost:

$$\frac{C(x)}{x} = C'(x).$$

(b) Interpret the conclusion of (a) geometrically in the figu

(c) If the average cost equals the marginal cost for some does x necessarily minimize the average cost?

Suppose that the fraction of individuals in the population infected with the virus is p, so the fraction uninfected is $q = 1 - p$. The probability that a given individual is unaffected is q, so the probability that all x individuals in a group are unaffected is q^x. Therefore, the probability that a pooled sample is infected is $1 - q^x$. Each group requires one test, and the infected groups require an extra x tests. Therefore the expected total number of tests to be performed is

$$T = \frac{N}{x} + \frac{N}{x}(1 - q^x)x = N\left(\frac{1}{x} + 1 - q^x\right).$$

For example, if $p = 0.01$, so that $q = 0.99$ and $x = 20$, then the expected number of tests required is $T = 0.23N$, a reduction of over 75%. But maybe we can do better by making a different choice for x.

(a) For $q = 0.99$, find the number x of samples in a group that minimizes T (i.e., solve $dT/dx = 0$). Show that the minimizing value of x satisfies

$$x = \frac{(0.99)^{-x/2}}{\sqrt{-\ln(0.99)}}.$$

(b) Use the technique of fixed-point iteration (see Section 4.2) to solve the equation in (a) for x. Start with $x = 20$, say.

4. (Measuring variations in g) The period P of a pendulum of length L is given by

$$P = 2\pi\sqrt{L/g},$$

where g is the acceleration of gravity.

(a) Assuming that L remains fixed, show that a 1% increase in g results in approximately a 0.5% decrease in the period P. (Variations in the period of a pendulum can be used to detect small variations in g from place to place on the earth's surface.)

(b) For fixed g, what percentage change in L will produce a 1% increase in P?

5. (Torricelli's Law) The rate at which a tank drains is proportional to the square root of the depth of liquid in the tank above the level of the drain: if $V(t)$ is the volume of liquid in the tank at time t, and $y(t)$ is the height of the surface of the liquid above the drain, then $dV/dt = -k\sqrt{y}$, where k is a constant depending on the size of the drain. For a cylindrical tank with constant cross-sectional area A with drain at the bottom:

(a) Verify that the depth $y(t)$ of liquid in the tank at time t satisfies $dy/dt = -(k/A)\sqrt{y}$.

(b) Verify that if the depth of liquid in the tank at $t = 0$ is y_0, then the depth at subsequent times during the draining process is $y = \left(\sqrt{y_0} - \dfrac{kt}{2A}\right)^2$.

(c) If the tank drains completely in time T, express the depth $y(t)$ at time t in terms of y_0 and T.

(d) In terms of T, how long does it take for half the liquid in the tank to drain out?

6. If a conical tank with top radius R and depth H drains according to Torricelli's Law and empties in time T, show that the depth of liquid in the tank at time t $(0 < t < T)$ is

$$y = y_0\left(1 - \frac{t}{T}\right)^{2/5},$$

where y_0 is the depth at $t = 0$.

7. Find the largest possible area of a right-angled triangle whose perimeter is P.

8. Find a tangent to the graph of $y = x^3 + ax^2 + bx + c$ that is not parallel to any other tangent.

9. (Branching angles for electric wires and pipes)

(a) The resistance offered by a wire to the flow of electric current through it is proportional to its length and inversely proportional to its cross-sectional area. Thus, the resistance R of a wire of length L and radius r is $R = kL/r^2$, where k is a positive constant. A long straight wire of length L and radius r_1 extends from A to B. A second straight wire of smaller radius r_2 is to be connected between a point P on AB and a point C at distance h from B such that CB is perpendicular to AB. (See Figure 4.72.) Find the value of the angle $\theta = \angle BPC$ that minimizes the total resistance of the path APC, that is, the resistance of AP plus the resistance of PC.

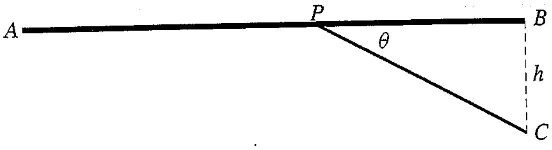

Figure 4.72

(b) The resistance of a pipe (e.g., a blood vessel) to the flow of liquid through it is, by Poiseuille's Law, proportional to its length and inversely proportional to the *fourth power* of its radius: $R = kL/r^4$. If the situation in part (a) represents pipes instead of wires, find the value of θ that minimizes the total resistance of the path APC. How does your answer relate to the answer for part (a)? Could you have predicted this relationship?

10. (The range of a spurt) A cylindrical water tank sitting on a horizontal table has a small hole located on its vertical wall at height h above the bottom of the tank. Water escapes from the tank horizontally through the hole and then curves down under the influence of gravity to strike the table at a distance R from the base of the tank, as shown in Figure 4.73. (We ignore air resistance.) Torricelli's Law implies that the speed v at which water escapes through the hole is proportional to the square root of the depth of the hole below the surface of the water: if the depth of water in the tank at time t is $y(t) > h$, then $v = k\sqrt{y - h}$, where the constant k depends on the size of the hole.

(a) Find the range R in terms of v and h.

(b) For a given depth y of water in the tank, how high should the hole be to maximize R?

(c) Suppose that the depth of water in the tank at time $t = 0$ is y_0, that the range R of the spurt is R_0 at that time, and that the water level drops to the height h of the hole in T minutes. Find, as a function of t, the range R of the water that escaped through the hole at time t.

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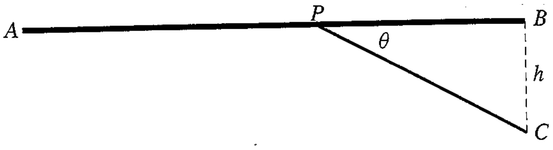

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