EXERCISES 3.1

Show that the functions f in Exercises 1–12 are one-to-one, and calculate the inverse functions f^{-1}. Specify the domains and ranges of f and f^{-1}.

1. $f(x) = x - 1$

2. $f(x) = 2x - 1$

3. $f(x) = \sqrt{x-1}$

4. $f(x) = -\sqrt{x-1}$

5. $f(x) = x^3$

6. $f(x) = 1 + \sqrt[3]{x}$

7. $f(x) = x^2, \quad x \le 0$

8. $f(x) = (1 - 2x)^3$

9. $f(x) = \dfrac{1}{x+1}$

10. $f(x) = \dfrac{x}{1+x}$

11. $f(x) = \dfrac{1-2x}{1+x}$

12. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

In Exercises 13–20, f is a one-to-one function with inverse f^{-1}. Calculate the inverses of the given functions in terms of f^{-1}.

13. $g(x) = f(x) - 2$

14. $h(x) = f(2x)$

15. $k(x) = -3f(x)$

16. $m(x) = f(x - 2)$

17. $p(x) = \dfrac{1}{1+f(x)}$

18. $q(x) = \dfrac{f(x) - 3}{2}$

19. $r(x) = 1 - 2f(3 - 4x)$

20. $s(x) = \dfrac{1+f(x)}{1-f(x)}$

In Exercises 21–23, show that the given function is one-to-one and find its inverse.

21. $f(x) = \begin{cases} x^2+1 & \text{if } x \ge 0 \\ x+1 & \text{if } x < 0 \end{cases}$

22. $g(x) = \begin{cases} x^3 & \text{if } x \ge 0 \\ x^{1/3} & \text{if } x < 0 \end{cases}$

23. $h(x) = x|x| + 1$

24. Find $f^{-1}(2)$ if $f(x) = x^3 + x$.

25. Find $g^{-1}(1)$ if $g(x) = x^3 + x - 9$.

26. Find $h^{-1}(-3)$ if $h(x) = x|x| + 1$.

27. Assume that the function $f(x)$ satisfies $f'(x) = \dfrac{1}{x}$ and that f is one-to-one. If $y = f^{-1}(x)$, show that $dy/dx = y$.

28. Find $\left(f^{-1}\right)'(x)$ if $f(x) = 1 + 2x^3$.

29. Show that $f(x) = \dfrac{4x^3}{x^2+1}$ has an inverse and find $\left(f^{-1}\right)'(2)$.

30. Find $\left(f^{-1}\right)'(-2)$ if $f(x) = x\sqrt{3+x^2}$.

31. If $f(x) = x^2/(1+\sqrt{x})$, find $f^{-1}(2)$ correct to 5 decimal places.

32. If $g(x) = 2x + \sin x$, show that g is invertible, and find $g^{-1}(2)$ and $(g^{-1})'(2)$ correct to 5 decimal places.

33. Show that $f(x) = x \sec x$ is one-to-one on $(-\pi/2, \pi/2)$. What is the domain of $f^{-1}(x)$? Find $(f^{-1})'(0)$.

34. If f and g have respective inverses f^{-1} and g^{-1}, show that the composite function $f \circ g$ has inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

35. For what values of the constants a, b, and c is the function $f(x) = (x - a)/(bx - c)$ self-inverse?

36. Can an even function be self-inverse? an odd function?

37. In this section it was claimed that an increasing (or decreasing) function defined on a single interval is necessarily one-to-one. Is the converse of this statement true? Explain.

38. Repeat Exercise 37 with the added assumption that f is continuous on the interval where it is defined.

3.2 Exponential and Logarithmic Functions

To begin we review exponential and logarithmic functions as you may have encountered them in your previous mathematical studies. In the following sections we will approach these functions from a different point of view and learn how to find their derivatives.

Exponentials

An **exponential function** is a function of the form $f(x) = a^x$, where the **base** a is a positive constant and the **exponent** x is the variable. Do not confuse such functions with **power** functions such as $f(x) = x^a$, where the base is variable and the exponent is constant. The exponential function a^x can be defined for integer and rational exponents x as follows:

If $a > 1$, then $\lim_{x\to 0+} \log_a x = -\infty$ and $\lim_{x\to\infty} \log_a x = \infty$.

If $0 < a < 1$, then $\lim_{x\to 0+} \log_a x = \infty$ and $\lim_{x\to\infty} \log_a x = -\infty$.

EXERCISES 3.2

Simplify the expressions in Exercises 1–18.

1. $\dfrac{3^3}{\sqrt{3^5}}$

2. $2^{1/2}8^{1/2}$

3. $\left(x^{-3}\right)^{-2}$

4. $\left(\dfrac{1}{2}\right)^x 4^{x/2}$

5. $\log_5 125$

6. $\log_4 \left(\dfrac{1}{8}\right)$

7. $\log_{1/3} 3^{2x}$

8. $2^{\log_4 8}$

9. $10^{-\log_{10}(1/x)}$

10. $x^{1/(\log_a x)}$

11. $(\log_a b)(\log_b a)$

12. $\log_x \left(x(\log_y y^2)\right)$

13. $(\log_4 16)(\log_4 2)$

14. $\log_{15} 75 + \log_{15} 3$

15. $\log_6 9 + \log_6 4$

16. $2\log_3 12 - 4\log_3 6$

17. $\log_a(x^4 + 3x^2 + 2) + \log_a(x^4 + 5x^2 + 6) -4\log_a \sqrt{x^2+2}$

18. $\log_\pi(1 - \cos x) + \log_\pi(1 + \cos x) - 2\log_\pi \sin x$

Use the base 10 exponential and logarithm functions 10^x and $\log x$ ($= \log_{10} x$) on a scientific calculator to evaluate the expressions or solve the equations in Exercises 19–24.

19. $3^{\sqrt{2}}$

20. $\log_3 5$

21. $2^{2x} = 5^{x+1}$

22. $x^{\sqrt{2}} = 3$

23. $\log_x 3 = 5$

24. $\log_3 x = 5$

Use the laws of exponents to prove the laws of logarithms in Exercises 25–28.

25. $\log_a \left(\dfrac{1}{x}\right) = -\log_a x$

26. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

27. $\log_a(x^y) = y\log_a x$

28. $\log_a x = (\log_b x)/(\log_b a)$

29. Solve $\log_4(x+4) - 2\log_4(x+1) = \dfrac{1}{2}$ for x.

30. Solve $2\log_3 x + \log_9 x = 10$ for x.

Evaluate the limits in Exercises 31–34.

31. $\lim_{x\to\infty} \log_x 2$

32. $\lim_{x\to 0+} \log_x (1/2)$

33. $\lim_{x\to 1+} \log_x 2$

34. $\lim_{x\to 1-} \log_x 2$

35. Suppose that $f(x) = a^x$ is differentiable at $x = 0$ and that $f'(0) = k$, where $k \neq 0$. Prove that f is differentiable at any real number x and that

$$f'(x) = k\,a^x = k\,f(x).$$

36. Continuing Exercise 35, prove that $f^{-1}(x) = \log_a x$ is differentiable at any $x > 0$ and that

$$(f^{-1})'(x) = \frac{1}{kx}.$$

3.3 The Natural Logarithm and Exponential

In this section we are going to define a function $\ln x$, called the *natural* logarithm of x, in a way that does not at first seem to have anything to do with the logarithms considered in Section 3.2. We will show, however, that it has the same properties as those logarithms, and in the end we will see that $\ln x = \log_e x$, the logarithm of x to a certain specific base e. We will show that $\ln x$ is a one-to-one function, defined for all positive real numbers. It must therefore have an inverse, e^x, that we will call *the* exponential function. Our final goal is to arrive at a definition of the exponential functions a^x (for any $a > 0$) that is valid for any real number x instead of just rational numbers, and that is known to be continuous and even differentiable without our having to assume those properties as we did in Section 3.2.

Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$. Thus,

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right)$$

$$= \frac{(x+2)(x+3)}{x+4} + \frac{(x+1)(x+3)}{x+4} + \frac{(x+1)(x+2)}{x+4} - \frac{(x+1)(x+2)(x+3)}{(x+4)^2}.$$

EXAMPLE 10 Find $\left.\frac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

$$\ln u = \frac{1}{2}\left(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\right)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1}\right).$$

At $x = 1$ we have $u = \sqrt{8} = 2\sqrt{2}$. Hence,

$$\left.\frac{du}{dx}\right|_{x=1} = \sqrt{2}\left(\frac{1}{2} + 1 + \frac{3}{2}\right) = 3\sqrt{2}.$$

EXERCISES 3.3

Simplify the expressions given in Exercises 1–10.

1. $e^3/\sqrt{e^5}$

2. $\ln\left(e^{1/2}e^{2/3}\right)$

3. $e^{5\ln x}$

4. $e^{(3\ln 9)/2}$

5. $\ln\frac{1}{e^{3x}}$

6. $e^{2\ln\cos x} + \left(\ln e^{\sin x}\right)^2$

7. $3\ln 4 - 4\ln 3$

8. $4\ln\sqrt{x} + 6\ln(x^{1/3})$

9. $2\ln x + 5\ln(x-2)$

10. $\ln(x^2 + 6x + 9)$

Solve the equations in Exercises 11–14 for x.

11. $2^{x+1} = 3^x$

12. $3^x = 9^{1-x}$

13. $\frac{1}{2^x} = \frac{5}{8^{x+3}}$

14. $2^{x^2-3} = 4^x$

Find the domains of the functions in Exercises 15–16.

15. $\ln\frac{x}{2-x}$

16. $\ln(x^2 - x - 2)$

Solve the inequalities in Exercises 17–18.

17. $\ln(2x-5) > \ln(7-2x)$

18. $\ln(x^2-2) \le \ln x$

In Exercises 19–48, differentiate the given functions. If possible, simplify your answers.

19. $y = e^{5x}$

20. $y = xe^x - x$

21. $y = \frac{x}{e^{2x}}$

22. $y = x^2e^{x/2}$

23. $y = \ln(3x-2)$

24. $y = \ln|3x-2|$

25. $y = \ln(1+e^x)$

26. $f(x) = e^{(x^2)}$

27. $y = \frac{e^x + e^{-x}}{2}$

28. $x = e^{3t}\ln t$

29. $y = e^{(e^x)}$

30. $y = \frac{e^x}{1+e^x}$

31. $y = e^x\sin x$

32. $y = e^{-x}\cos x$

33. $y = \ln\ln x$

34. $y = x\ln x - x$

35. $y = x^2\ln x - \frac{x^2}{2}$

36. $y = \ln|\sin x|$

37. $y = 5^{2x+1}$

38. $y = 2^{(x^2-3x+8)}$

39. $g(x) = t^x x^t$

40. $h(t) = t^x - x^t$

41. $f(s) = \log_a(bs+c)$

42. $g(x) = \log_x(2x+3)$

43. $y = x^{\sqrt{x}}$

44. $y = (1/x)^{\ln x}$

45. $y = \ln|\sec x + \tan x|$

46. $y = \ln|x + \sqrt{x^2 - a^2}|$

47. $y = \ln\left(\sqrt{x^2+a^2} - x\right)$

48. $y = (\cos x)^x - x^{\cos x}$

49. Find the nth derivative of $f(x) = xe^{ax}$.

50. Show that the nth derivative of $(ax^2 + bx + c)e^x$ is a function of the same form but with different constants.

51. Find the first four derivatives of e^{x^2}.

52. Find the nth derivative of $\ln(2x + 1)$.

53. Differentiate (a) $f(x) = (x^x)^x$ and (b) $g(x) = x^{(x^x)}$. Which function grows more rapidly as x grows large?

54. Solve the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = a$, where $a > 0$. The exponent tower goes on forever.

Use logarithmic differentiation to find the required derivatives in Exercises 55–57.

55. $f(x) = (x-1)(x-2)(x-3)(x-4)$. Find $f'(x)$.

56. $F(x) = \dfrac{\sqrt{1+x}(1-x)^{1/3}}{(1+5x)^{4/5}}$. Find $F'(0)$.

57. $f(x) = \dfrac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}$. Find $f'(2)$. Also find $f'(1)$.

58. At what points does the graph $y = x^2 e^{-x^2}$ have a horizontal tangent line?

59. Let $f(x) = xe^{-x}$. Determine where f is increasing and where it is decreasing. Sketch the graph of f.

60. Find the equation of a straight line of slope 4 that is tangent to the graph of $y = \ln x$.

61. Find an equation of the straight line tangent to the curve $y = e^x$ and passing through the origin.

62. Find an equation of the straight line tangent to the curve $y = \ln x$ and passing through the origin.

63. Find an equation of the straight line that is tangent to $y = 2^x$ and that passes through the point $(1, 0)$.

64. For what values of $a > 0$ does the curve $y = a^x$ intersect the straight line $y = x$?

65. Find the slope of the curve $e^{xy} \ln \dfrac{x}{y} = x + \dfrac{1}{y}$ at $(e, 1/e)$.

66. Find an equation of the straight line tangent to the curve $xe^y + y - 2x = \ln 2$ at the point $(1, \ln 2)$.

67. Find the derivative of $f(x) = Ax \cos \ln x + Bx \sin \ln x$. Use the result to help you find the indefinite integrals $\int \cos \ln x \, dx$ and $\int \sin \ln x \, dx$.

68. Let $F_{A,B}(x) = Ae^x \cos x + Be^x \sin x$. Show that $(d/dx)F_{A,B}(x) = F_{A+B,B-A}(x)$.

69. Using the results of Exercise 68, find (a) $(d^2/dx^2)F_{A,B}(x)$ and (b) $(d^3/dx^3)e^x \cos x$.

70. Find $\dfrac{d}{dx}(Ae^{ax}\cos bx + Be^{ax}\sin bx)$ and use the answer to help you evaluate (a) $\int e^{ax}\cos bx\,dx$ and (b) $\int e^{ax}\sin bx\,dx$.

71. Prove identity (ii) of Theorem 2 by examining the derivative of the left side minus the right side as was done in the proof of identity (i).

72. Deduce identity (iii) of Theorem 2 from identities (i) and (ii).

73. Prove identity (iv) of Theorem 2 for rational exponents r by the same method used for Exercise 71.

74. Let $x > 0$, and let $F(x)$ be the area bounded by the curve $y = t^2$, the t-axis, and the vertical lines $t = 0$ and $t = x$. Using the method of the proof of Theorem 1, show that $F'(x) = x^2$. Hence, find an explicit formula for $F(x)$. What is the area of the region bounded by $y = t^2$, $y = 0$, $t = 0$, and $t = 2$?

75. Carry out the following steps to show that $2 < e < 3$. Let $f(t) = 1/t$ for $t > 0$.
   (a) Show that the area under $y = f(t)$, above $y = 0$, and between $t = 1$ and $t = 2$ is less than 1 square unit. Deduce that $e > 2$.
   (b) Show that all tangent lines to the graph of f lie below the graph. *Hint:* $f''(t) = 2/t^3 > 0$.
   (c) Find the lines T_2 and T_3 that are tangent to $y = f(t)$ at $t = 2$ and $t = 3$, respectively.
   (d) Find the area A_2 under T_2, above $y = 0$, and between $t = 1$ and $t = 2$. Also find the area A_3 under T_3, above $y = 0$, and between $t = 2$ and $t = 3$.
   (e) Show that $A_2 + A_3 > 1$ square unit. Deduce that $e < 3$.

3.4 Growth and Decay

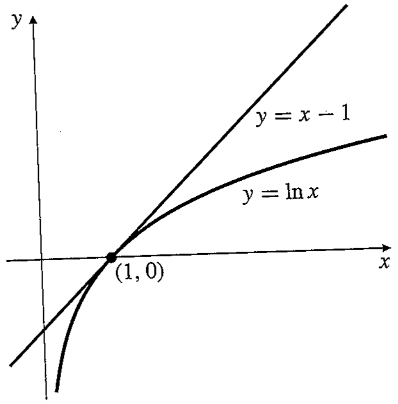

Figure 3.14 $\ln x \le x - 1$ for $x > 0$

In this section we will study the use of exponential functions to model the growth rates of quantities whose rate of growth is directly related to their size. The growth of such quantities is typically governed by differential equations whose solutions involve exponential functions. Before delving into this topic, we prepare the way by examining the growth behaviour of exponential and logarithmic functions.

The Growth of Exponentials and Logarithms

In Section 3.3 we showed that both e^x and $\ln x$ grow large (approach infinity) as x grows large. However, e^x increases very rapidly as x increases, and $\ln x$ increases very slowly. In fact, e^x increases faster than any positive power of x (no matter how large the power), while $\ln x$ increases more slowly than any positive power of x (no matter how small the power). To verify this behaviour we start with an inequality satisfied by $\ln x$. The straight line $y = x - 1$ is tangent to the curve $y = \ln x$ at the point $(1, 0)$. The following theorem asserts that the curve lies below that line. (See Figure 3.14.)

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right),$$

which is called the **logistic equation** since it models growth that is limited by the *supply* of necessary resources. Observe that $dy/dt > 0$ if $0 < y < L$ and that this rate is small if y is small (there are few rabbits to reproduce) or if y is close to L (there are almost as many rabbits as the available resources can feed). Observe also that $dy/dt < 0$ if $y > L$; there being more animals than the resources can feed, the rabbits die at a greater rate than they are born. Of course, the steady-state populations $y = 0$ and $y = L$ are solutions of the logistic equation; for both of these $dy/dt = 0$. We will examine techniques for solving differential equations like the logistic equation in Section 7.9. For now, we invite the reader to verify by differentiation that the solution satisfying $y(0) = y_0$ is

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}}.$$

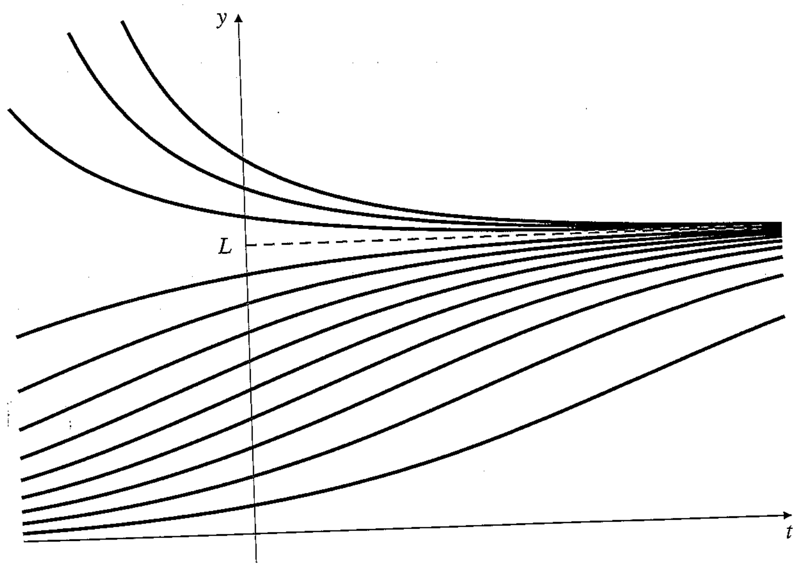

Figure 3.16 Some logistic curves

Observe that, as expected, if $0 < y_0 < L$, then

$$\lim_{t\to\infty} y(t) = L, \qquad \lim_{t\to-\infty} y(t) = 0.$$

The solution given above also holds for $y_0 > L$. However, the solution does not approach 0 as t approaches $-\infty$ in this case. It has a vertical asymptote at a certain negative value of t. (See Exercise 30 below.) The graphs of solutions of the logistic equation for various positive values of y_0 are given in Figure 3.16.

EXERCISES 3.4

Evaluate the limits in Exercises 1–8.

1. $\lim_{x\to\infty} x^3 e^{-x}$

2. $\lim_{x\to\infty} x^{-3} e^x$

3. $\lim_{x\to\infty} \frac{2e^x - 3}{e^x + 5}$

4. $\lim_{x\to\infty} \frac{x - 2e^{-x}}{x + 3e^{-x}}$

5. $\lim_{x\to 0+} x \ln x$

6. $\lim_{x\to 0+} \frac{\ln x}{x}$

7. $\lim_{x\to 0} x\left(\ln|x|\right)^2$

8. $\lim_{x\to\infty} \frac{(\ln x)^3}{\sqrt{x}}$

9. **(Bacterial growth)** Bacteria grow in a certain culture at a rate proportional to the amount present. If there are 100 bacteria present initially and the amount doubles in 1 h, how many will there be after a further $1\frac{1}{2}$ h?

10. **(Dissolving sugar)** Sugar dissolves in water at a rate proportional to the amount still undissolved. If there were 50 kg of sugar present initially, and at the end of 5 h only 20 kg are left, how much longer will it take until 90% of the sugar is disssolved?

11. **(Radioactive decay)** A radioactive substance decays at a rate proportional to the amount present. If 30% of such a substance decays in 15 years, what is the half-life of the substance?

12. **(Half-life of radium)** If the half-life of radium is 1,690 years, what percentage of the amount present now will be remaining after (a) 100 years, (b) 1,000 years?

13. Find the half-life of a radioactive substance if after 1 year 99.57% of an initial amount still remains.

14. **(Bacterial growth)** In a certain culture where the rate of growth of bacteria is proportional to the number present, the number triples in 3 days. If at the end of 7 days there are 10 million bacteria present in the culture, how many were present initially?

15. **(Weight of a newborn)** In the first few weeks after birth, a baby gains weight at a rate proportional to its weight. A baby weighing 4 kg at birth weighs 4.4 kg after 2 weeks. How much did the baby weigh 5 days after birth?

16. **(Electric current)** When a simple electrical circuit containing inductance and resistance but no capacitance has the electromotive force removed, the rate of decrease of the current is proportional to the current. If the current is $I(t)$ amperes t s after cutoff, and if $I = 40$ when $t = 0$, and $I = 15$ when $t = 0.01$, find a formula for $I(t)$.

17. **(Continuously compounding interest)** How much money needs to be invested today at a nominal rate of 4% compounded continuously, in order that it should grow to $10,000 in 7 years?

18. **(Continuously compounding interest)** Money invested at compound interest (with instantaneous compounding) accumulates at a rate proportional to the amount present. If an initial investment of $1,000 grows to $1,500 in exactly 5 years, find (a) the doubling time for the investment and (b) the effective annual rate of interest being paid.

19. **(Purchasing power)** If the purchasing power of the dollar is decreasing at an effective rate of 9% annually, how long will it take for the purchasing power to be reduced to 25 cents?

20. **(Effective interest rate)** A bank claims to pay interest at an effective rate of 9.5% on an investment account. If the interest is actually being compounded monthly, what is the nominal rate of interest being paid on the account?

21. Suppose that 1,000 rabbits are introduced onto an island where they have no natural predators. During the next five years the rabbit population grows exponentially. After the first two years the population grew to 3,500 rabbits. After the first five years a rabbit virus is sprayed on the island and after that the rabbit population decays exponentially. Two years after the virus was introduced (so seven years after rabbits were introduced to the island) the rabbit population dropped to 3,000 rabbits. How many rabbits will there be on the island 10 years after they were introduced?

22. Lab rats are to be used in experiments on an isolated island. Initially R rats are brought to the island and released. Having a plentiful food supply and no natural predators on the island, the rat population grows exponentially and doubles in three months. At the end of the fifth month, and at the end of every five months thereafter, 1,000 of the rats are captured and killed. What is the minimum value of R that ensures that the scientists will never run out of rats?

Differential equations of the form $y' = a + by$

23. Suppose that $f(x)$ satisfies the differential equation

$$f'(x) = a + bf(x),$$

where a and b are constants.

(a) Solve the differential equation by substituting $u(x) = a + bf(x)$ and solving the simpler differential equation that results for $u(x)$.

(b) Solve the initial-value problem:

$$\begin{cases} \dfrac{dy}{dx} = a + by \\ y(0) = y_0 \end{cases}$$

24. **(Drug concentrations in the blood)** A drug is introduced into the bloodstream intravenously at a constant rate and breaks down and is eliminated from the body at a rate proportional to its concentration in the blood. The concentration $x(t)$ of the drug in the blood satisfies the differential equation

$$\frac{dx}{dt} = a - bx,$$

where a and b are positive constants.

(a) What is the limiting concentration $\lim_{t\to\infty} x(t)$ of the drug in the blood?

(b) Find the concentration of the drug in the blood at time t, given that the concentration was zero at $t = 0$.

(c) How long after $t = 0$ will it take for the concentration to rise to half its limiting value?

25. **(Cooling)** Use Newton's law of cooling to determine the reading on a thermometer 5 min after it is taken from an oven at 72 °C to the outdoors where the temperature is 20 °C, if the reading dropped to 48 °C after one min.

26. **(Cooling)** An object is placed in a freezer maintained at a temperature of −5 °C. If the object cools from 45 °C to 20 °C in 40 min, how many more minutes will it take to cool to 0 °C?

27. **(Warming)** If an object in a room warms up from 5 °C to 10 °C in 4 min, and if the room is being maintained at 20 °C, how much longer will the object take to warm up to 15 °C? Assume the object warms at a rate proportional to the difference between its temperature and room temperature.

The logistic equation

28. Suppose the quantity $y(t)$ exhibits logistic growth. If the values of $y(t)$ at times $t = 0$, $t = 1$, and $t = 2$ are y_0, y_1, and y_2, respectively, find an equation satisfied by the limiting value L of $y(t)$, and solve it for L. If $y_0 = 3$, $y_1 = 5$, and $y_2 = 6$, find L.

29. Show that a solution $y(t)$ of the logistic equation having $0 < y(0) < L$ is increasing most rapidly when its value is $L/2$. (*Hint:* You do not need to use the formula for the solution to see this.)

30. If $y_0 > L$, find the interval on which the given solution of the logistic equation is valid. What happens to the solution as t approaches the left endpoint of this interval?

31. If $y_0 < 0$, find the interval on which the given solution of the logistic equation is valid. What happens to the solution as t approaches the right endpoint of this interval?

32. (Modelling an epidemic) The number y of persons infected by a highly contagious virus is modelled by a logistic curve

$$y = \frac{L}{1 + Me^{-kt}},$$

where t is measured in months from the time the outbreak was discovered. At that time there were 200 infected persons, and the number grew to 1,000 after 1 month. Eventually, the number levelled out at 10,000. Find the values of the parameters L, M, and k of the model.

33. Continuing Exercise 32, how many people were infected 3 months after the outbreak was discovered, and how fast was the number growing at that time?

3.5 The Inverse Trigonometric Functions

The six trigonometric functions are periodic and, hence, not one-to-one. However, as we did with the function x^2 in Section 3.1, we can restrict their domains in such a way that the restricted functions are one-to-one and invertible.

The Inverse Sine (or Arcsine) Function

Let us define a function $\operatorname{Sin} x$ (note the capital letter) to be $\sin x$, restricted so that its domain is the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:

DEFINITION 8

The restricted function Sin x

$$\operatorname{Sin} x = \sin x \qquad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Since its derivative $\cos x$ is positive on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function $\operatorname{Sin} x$ is increasing on its domain, so it is a one-to-one function. It has domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$. (See Figure 3.17.)

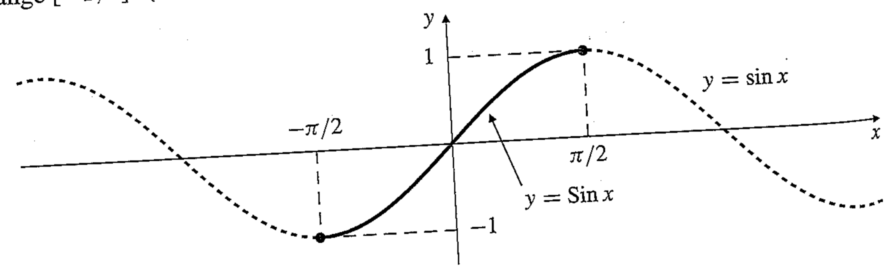

Figure 3.17 The graph of $\operatorname{Sin} x$ forms part of the graph of $\sin x$

Being one-to-one, Sin has an inverse function which is denoted $\sin^{-1}$ (or, in some books and computer programs, by arcsin, Arcsin, or asin) and which is called the **inverse sine** or **arcsine** function.

DEFINITION 9

The inverse sine function $\sin^{-1} x$ or $\arcsin x$

$$y = \sin^{-1} x \iff x = \operatorname{Sin} y$$
$$\iff x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

The graph of $\sin^{-1}$ is shown in Figure 3.18; it is the reflection of the graph of Sin in the line $y = x$. The domain of $\sin^{-1}$ is $[-1, 1]$ (the range of Sin), and the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the domain of Sin). The cancellation identities for Sin and $\sin^{-1}$ are

The corresponding integration formula takes different forms on intervals where $x \geq 1$ or $x \leq -1$:

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \begin{cases} \sec^{-1} x + C & \text{on intervals where } x \geq 1 \\ -\sec^{-1} x + C & \text{on intervals where } x \leq -1 \end{cases}$$

Finally, note that $\csc^{-1}$ and $\cot^{-1}$ are defined similarly to $\sec^{-1}$. They are seldom encountered.

DEFINITION 14

The inverse cosecant and inverse cotangent functions

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), \quad (|x| \geq 1); \qquad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), \quad (x \neq 0)$$

EXERCISES 3.5

In Exercises 1–12, evaluate the given expression.

1. $\sin^{-1} \frac{\sqrt{3}}{2}$
2. $\cos^{-1}\left(\frac{-1}{2}\right)$
3. $\tan^{-1}(-1)$
4. $\sec^{-1}\sqrt{2}$
5. $\sin(\sin^{-1} 0.7)$
6. $\cos(\sin^{-1} 0.7)$
7. $\tan^{-1}\left(\tan \frac{2\pi}{3}\right)$
8. $\sin^{-1}(\cos 40°)$
9. $\cos^{-1}(\sin(-0.2))$
10. $\sin\left(\cos^{-1}\left(\frac{-1}{3}\right)\right)$
11. $\cos\left(\tan^{-1}\frac{1}{2}\right)$
12. $\tan(\tan^{-1} 200)$

In Exercises 13–18, simplify the given expression.

13. $\sin(\cos^{-1} x)$
14. $\cos(\sin^{-1} x)$
15. $\cos(\tan^{-1} x)$
16. $\sin(\tan^{-1} x)$
17. $\tan(\cos^{-1} x)$
18. $\tan(\sec^{-1} x)$

In Exercises 19–32, differentiate the given function and simplify the answer whenever possible.

19. $y = \sin^{-1}\left(\frac{2x-1}{3}\right)$
20. $y = \tan^{-1}(ax+b)$
21. $y = \cos^{-1}\left(\frac{x-b}{a}\right)$
22. $f(x) = x\sin^{-1} x$
23. $f(t) = t\tan^{-1} t$
24. $u = z^2 \sec^{-1}(1+z^2)$
25. $F(x) = (1+x^2)\tan^{-1} x$
26. $y = \sin^{-1}\frac{a}{x}$
27. $G(x) = \frac{\sin^{-1} x}{\sin^{-1} 2x}$
28. $H(t) = \frac{\sin^{-1} t}{\sin t}$
29. $f(x) = (\sin^{-1} x^2)^{1/2}$
30. $y = \cos^{-1}\frac{a}{\sqrt{a^2+x^2}}$
31. $y = \sqrt{a^2-x^2} + a\sin^{-1}\frac{x}{a} \quad (a > 0)$
32. $y = a\cos^{-1}\left(1-\frac{x}{a}\right) - \sqrt{2ax-x^2} \quad (a > 0)$
33. Find the slope of the curve $\tan^{-1}\left(\frac{2x}{y}\right) = \frac{\pi x}{y^2}$ at the point $(1, 2)$.
34. Find equations of two straight lines tangent to the graph of $y = \sin^{-1} x$ and having slope 2.
35. Show that, on their respective domains, $\sin^{-1}$ and $\tan^{-1}$ are increasing functions and $\cos^{-1}$ is a decreasing function.
36. The derivative of $\sec^{-1} x$ is positive for every x in the domain of $\sec^{-1}$. Does this imply that $\sec^{-1}$ is increasing on its domain? Why?
37. Sketch the graph of $\csc^{-1} x$ and find its derivative.
38. Sketch the graph of $\cot^{-1} x$ and find its derivative.
39. Show that $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ for $x > 0$. What is the sum if $x < 0$?
40. Find the derivative of $g(x) = \tan(\tan^{-1} x)$ and sketch the graph of g.

In Exercises 41–44, plot the graphs of the given functions by first calculating and simplifying the derivative of the function. Where is each function continuous? Where is it differentiable?

41. $\cos^{-1}(\cos x)$
42. $\sin^{-1}(\cos x)$
43. $\tan^{-1}(\tan x)$
44. $\tan^{-1}(\cot x)$
45. Show that $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ if $|x| < 1$.
46. Show that $\sec^{-1} x = \begin{cases} \tan^{-1}\sqrt{x^2-1} & \text{if } x \geq 1 \\ \pi - \tan^{-1}\sqrt{x^2-1} & \text{if } x \leq -1 \end{cases}$
47. Show that $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ for all x.
48. Show that $\sec^{-1} x = \begin{cases} \sin^{-1}\frac{\sqrt{x^2-1}}{x} & \text{if } x \geq 1 \\ \pi - \sin^{-1}\frac{\sqrt{x^2-1}}{x} & \text{if } x \leq -1 \end{cases}$
49. Show that the function $f(x)$ of Example 9 is also constant on the interval $(-\infty, -1)$. Find the value of the constant. *Hint:* Find $\lim_{x\to-\infty} f(x)$.
50. Find the derivative of $f(x) = x - \tan^{-1}(\tan x)$. What does your answer imply about $f(x)$? Calculate $f(0)$ and $f(\pi)$. Is there a contradiction here?
51. Find the derivative of $f(x) = x - \sin^{-1}(\sin x)$ for $-\pi \leq x \leq \pi$ and sketch the graph of f on that interval.

In Exercises 52–55, solve the initial-value problems.

52. $\begin{cases} y' = \dfrac{1}{1+x^2} \\ y(0) = 1 \end{cases}$

53. $\begin{cases} y' = \dfrac{1}{9+x^2} \\ y(3) = 2 \end{cases}$

54. $\begin{cases} y' = \dfrac{1}{\sqrt{1-x^2}} \\ y(1/2) = 1 \end{cases}$

55. $\begin{cases} y' = \dfrac{4}{\sqrt{25-x^2}} \\ y(0) = 0 \end{cases}$

3.6 Hyperbolic Functions

Any function defined on the real line can be expressed (in a unique way) as the sum of an even function and an odd function. (See Exercise 35 of Section P.5.) The **hyperbolic functions** $\cosh x$ and $\sinh x$ are, respectively, the even and odd functions whose sum is the exponential function e^x.

DEFINITION 15

The hyperbolic cosine and hyperbolic sine functions

For any real x the **hyperbolic cosine**, $\cosh x$, and the **hyperbolic sine**, $\sinh x$, are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(The symbol "sinh" is somewhat hard to pronounce as written. Some people say "shine," and others say "sinch.") Recall that cosine and sine are called *circular functions* because, for any t, the point $(\cos t, \sin t)$ lies on the circle with equation $x^2 + y^2 = 1$. Similarly, cosh and sinh are called *hyperbolic functions* because the point $(\cosh t, \sinh t)$ lies on the rectangular hyperbola with equation $x^2 - y^2 = 1$,

$$\cosh^2 t - \sinh^2 t = 1 \quad \text{for any real } t.$$

To see this, observe that

$$\begin{aligned} \cosh^2 t - \sinh^2 t &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})\right) \\ &= \frac{1}{4}(2+2) = 1. \end{aligned}$$

There is no interpretation of t as an arc length or angle as there was in the circu case; however, the *area* of the *hyperbolic sector* bounded by $y = 0$, the hyperb $x^2 - y^2 = 1$, and the ray from the origin to $(\cosh t, \sinh t)$ is $t/2$ square units (s Exercise 21 of Section 8.4), just as is the area of the circular sector bounded by $y =$ the circle $x^2 + y^2 = 1$, and the ray from the origin to $(\cos t, \sin t)$. (See Figure 3.2

Observe that, similar to the corresponding values of $\cos x$ and $\sin x$, we have

$$\cosh 0 = 1 \quad \text{and} \quad \sinh 0 = 0,$$

and $\cosh x$, like $\cos x$, is an even function, and $\sinh x$, like $\sin x$, is an odd functio

$$\cosh(-x) = \cosh x, \qquad \sinh(-x) = -\sinh x.$$

The graphs of cosh and sinh are shown in Figure 3.27. The graph $y = \cosh x$ is c a **catenary**. A chain hanging by its ends will assume the shape of a catenary.

27. Solve $\begin{cases} y'' + y = 0 \\ y(2) = 3 \\ y'(2) = -4. \end{cases}$

28. Solve $\begin{cases} y'' + \omega^2 y = 0 \\ y(a) = A \\ y'(a) = B. \end{cases}$

29. What mass should be suspended from the spring in Example 6 to provide a system whose natural frequency of oscillation is 10 Hz? Find the displacement of such a mass from its equilibrium position t s after it is pulled down 1 cm from equilibrium and flicked upward with a speed of 2 cm/s. What is the amplitude of this motion?

30. A mass of 400 g suspended from a certain elastic spring will oscillate with a frequency of 24 Hz. What would be the frequency if the 400 g mass were replaced with a 900 g mass? a 100 g mass?

31. Show that if t_0, A, and B are constants and $k = -b/(2a)$ and $\omega = \sqrt{4ac - b^2}/(2a)$, then

$$y = e^{kt}\left[A\cos\left(\omega(t-t_0)\right) + B\sin\left(\omega(t-t_0)\right)\right]$$

is an alternative to the general solution for $ay'' + by' + cy = 0$ for Case III ($b^2 - 4ac < 0$). This form of the general solution is useful for solving initial-value problems where $y(t_0)$ and $y'(t_0)$ are specified.

32. Show that if t_0, A, and B are constants and $k = -b/(2a)$ and $\omega = \sqrt{b^2 - 4ac}/(2a)$, then

$$y = e^{kt}\left[A\cosh\left(\omega(t-t_0)\right) + B\sinh\left(\omega(t-t_0)\right)\right]$$

is an alternative to the general solution for $ay'' + by' + cy = 0$ for Case I ($b^2 - 4ac > 0$). This form of the general solution is useful for solving initial-value problems where $y(t_0)$ and $y'(t_0)$ are specified.

Use the forms of solution provided by the previous two exercises to solve the initial-value problems in Exercises 33–34.

33. $\begin{cases} y'' + 2y' + 5y = 0 \\ y(3) = 2 \\ y'(3) = 0. \end{cases}$

34. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(3) = 1 \\ y'(3) = 0. \end{cases}$

35. By using the change of dependent variable $u(x) = c - k^2 y(x)$, solve the initial-value problem

$$\begin{cases} y''(x) = c - k^2 y(x) \\ y(0) = a \\ y'(0) = b. \end{cases}$$

36. A mass is attached to a spring mounted horizontally so the mass can slide along the top of a table. With a suitable choice of units, the position $x(t)$ of the mass at time t is governed by the differential equation

$$x'' = -x + F,$$

where the $-x$ term is due to the elasticity of the spring, and the F is due to the friction of the mass with the table. The frictional force should be constant in magnitude and directed opposite to the velocity of the mass when the mass is moving. When the mass is stopped, the friction should be constant and opposed to the spring force unless the spring force has the smaller magnitude, in which case the friction force should just cancel the spring force and the mass should remain at rest thereafter. For this problem, let the magnitude of the friction force be 1/5. Accordingly,

$$F = \begin{cases} -\frac{1}{5} & \text{if } x' > 0 \text{ or if } x' = 0 \text{ and } x < -\frac{1}{5} \\ \frac{1}{5} & \text{if } x' < 0 \text{ or if } x' = 0 \text{ and } x > \frac{1}{5} \\ x & \text{if } x' = 0 \text{ and } |x| \le \frac{1}{5}. \end{cases}$$

Find the position $x(t)$ of the mass at all times $t > 0$ if $x(0) = 1$ and $x'(0) = 0$.

CHAPTER REVIEW

Key Ideas

- **State the laws of exponents.**
- **State the laws of logarithms.**
- **What is the significance of the number e?**
- **What do the following statements and phrases mean?**
  - ◇ f is one-to-one. ◇ f is invertible.
  - ◇ Function f^{-1} is the inverse of function f.
  - ◇ $a^b = c$ ◇ $\log_a b = c$
  - ◇ the natural logarithm of x
  - ◇ logarithmic differentiation
  - ◇ the half-life of a varying quantity
  - ◇ The quantity y exhibits exponential growth.
  - ◇ The quantity y exhibits logistic growth.
  - ◇ $y = \sin^{-1} x$ ◇ $y = \tan^{-1} x$
  - ◇ The quantity y exhibits simple harmonic motion.
  - ◇ The quantity y exhibits damped harmonic motion.
- **Define the functions $\sinh x$, $\cosh x$, and $\tanh x$.**
- **What kinds of functions satisfy second-order differential equations with constant coefficients?**

Review Exercises

1. If $f(x) = 3x + x^3$, show that f has an inverse and find the slope of $y = f^{-1}(x)$ at $x = 0$.

2. Let $f(x) = \sec^2 x \tan x$. Show that f is increasing on the interval $(-\pi/2, \pi/2)$ and, hence, one-to-one and invertible there. What is the domain of f^{-1}? Find $(f^{-1})'(2)$. *Hint:* $f(\pi/4) = 2$.

Exercises 3–5 refer to the function $f(x) = x\,e^{-x^2}$.

3. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
4. On what intervals is f increasing? decreasing?
5. What are the maximum and minimum values of $f(x)$?
6. Find the points on the graph of $y = e^{-x}\sin x$, $(0 \le x \le 2\pi)$, where the graph has a horizontal tangent line.
7. Suppose that a function $f(x)$ satisfies $f'(x) = x\,f(x)$ for all real x, and $f(2) = 3$. Calculate the derivative of $f(x)/e^{x^2/2}$, and use the result to help you find $f(x)$ explicitly.
8. A lump of modelling clay is being rolled out so that it maintains the shape of a circular cylinder. If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself.
9. (a) What nominal interest rate, compounded continuously, will cause an investment to double in 5 years?

   (b) By about how many days will the doubling time in part (a) increase if the nominal interest rate drops by 0.5%?
10. **(A poor man's natural logarithm)**

    (a) Show that if $a > 0$, then
    $$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$
    Hence show that
    $$\lim_{n\to\infty} n(a^{1/n} - 1) = \ln a.$$
    (b) Most calculators, even nonscientific ones, have a square root key. If n is a power of 2, say $n = 2^k$, then $a^{1/n}$ can be calculated by entering a and hitting the square root key k times:
    $$a^{1/2^k} = \sqrt{\sqrt{\cdots\sqrt{a}}} \qquad (k \text{ square roots}).$$
    Then you can subtract 1 and multiply by n to get an approximation for $\ln a$. Use $n = 2^{10} = 1024$ and $n = 2^{11} = 2048$ to find approximations for $\ln 2$. Based on the agreement of these two approximations, quote a value of $\ln 2$ to as many decimal places as you feel justified.
11. A nonconstant function f satisfies
    $$\frac{d}{dx}\big(f(x)\big)^2 = \big(f'(x)\big)^2$$
    for all x. If $f(0) = 1$, find $f(x)$.
12. If $f(x) = (\ln x)/x$, show that $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $x > e$, so that $f(x)$ has a maximum value at $x = e$. Use this to show that $e^\pi > \pi^e$.
13. Find an equation of a straight line that passes through the origin and is tangent to the curve $y = x^x$.
14. (a) Find $x \ne 2$ such that $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$.

    (b) Find $b > 1$ such that there is *no* $x \ne b$ with $\dfrac{\ln x}{x} = \dfrac{\ln b}{b}$.
15. Investment account A bears simple interest at a certain rate. Investment account B bears interest at the same nominal rate but compounded instantaneously. If \$1,000 is invested in each account, B produces \$10 more in interest after one year than does A. Find the nominal rate both accounts use.
16. Express each of the functions $\cos^{-1} x$, $\cot^{-1} x$, and $\csc^{-1} x$ in terms of $\tan^{-1}$.
17. Express each of the functions $\cos^{-1} x$, $\cot^{-1} x$, and $\csc^{-1} x$ in terms of $\sin^{-1}$.
18. **(A warming problem)** A bottle of milk at 5 °C is removed from a refrigerator into a room maintained at 20 °C. After 12 min the temperature of the milk is 12 °C. How much longer will it take for the milk to warm up to 18 °C?
19. **(A cooling problem)** A kettle of hot water at 96 °C is allowed to sit in an air-conditioned room. The water cools to 60 °C in 10 min and then to 40 °C in another 10 min. What is the temperature of the room?
20. Show that $e^x > 1 + x$ if $x \ne 0$.
21. Use mathematical induction to show that
    $$e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$
    if $x > 0$ and n is any positive integer.

Challenging Problems

1. (a) Show that the function $f(x) = x^x$ is strictly increasing on $[e^{-1}, \infty)$.

   (b) If g is the inverse function to f of part (a), show that
   $$\lim_{y\to\infty} \frac{g(y)\ln(\ln y)}{\ln y} = 1$$
   Hint: Start with the equation $y = x^x$ and take the ln of both sides twice.

Two models for incorporating air resistance into the analysis of the motion of a falling body

2. **(Air resistance proportional to speed)** An object falls under gravity near the surface of the earth, and its motion is impeded by air resistance proportional to its speed. Its velocity v therefore satisfies the equation
   $$\frac{dv}{dt} = -g - kv, \qquad (*)$$
   where k is a positive constant depending on such factors as the shape and density of the object and the density of the air.

   (a) Find the velocity of the object as a function of time t, given that it was v_0 at $t = 0$.

   (b) Find the limiting velocity $\lim_{t\to\infty} v(t)$. Observe that this can be done either directly from $(*)$ or from the solution found in (a).

   (c) If the object was at height y_0 at time $t = 0$, find its height $y(t)$ at any time during its fall.
3. **(Air resistance proportional to the square of speed)** Under certain conditions a better model for the effect of air resistance on a moving object is one where the resistance is proportional to the square of the speed. For an object falling under constant gravitational acceleration g, the equation of motion is
   $$\frac{dv}{dt} = -g - kv|v|,$$