EXERCISES 2.1

In Exercises 1–12, find an equation of the straight line tangent to the given curve at the point indicated.

1. $y = 3x - 1$ at $(1, 2)$
2. $y = x/2$ at $(a, a/2)$
3. $y = 2x^2 - 5$ at $(2, 3)$
4. $y = 6 - x - x^2$ at $x = -2$
5. $y = x^3 + 8$ at $x = -2$
6. $y = \dfrac{1}{x^2+1}$ at $(0, 1)$
7. $y = \sqrt{x+1}$ at $x = 3$
8. $y = \dfrac{1}{\sqrt{x}}$ at $x = 9$
9. $y = \dfrac{2x}{x+2}$ at $x = 2$
10. $y = \sqrt{5 - x^2}$ at $x = 1$
11. $y = x^2$ at $x = x_0$
12. $y = \dfrac{1}{x}$ at $\left(a, \dfrac{1}{a}\right)$

Do the graphs of the functions f in Exercises 13–17 have tangent lines at the given points? If yes, what is the tangent line?

13. $f(x) = \sqrt{|x|}$ at $x = 0$
14. $f(x) = (x-1)^{4/3}$ at $x = 1$
15. $f(x) = (x+2)^{3/5}$ at $x = -2$
16. $f(x) = |x^2 - 1|$ at $x = 1$
17. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$ at $x = 0$
18. Find the slope of the curve $y = x^2 - 1$ at the point $x = x_0$. What is the equation of the tangent line to $y = x^2 - 1$ that has slope -3?
19. (a) Find the slope of $y = x^3$ at the point $x = a$.
    (b) Find the equations of the straight lines having slope 3 that are tangent to $y = x^3$.
20. Find all points on the curve $y = x^3 - 3x$ where the tangent line is parallel to the x-axis.
21. Find all points on the curve $y = x^3 - x + 1$ where the tangent line is parallel to the line $y = 2x + 5$.
22. Find all points on the curve $y = 1/x$ where the tangent line is perpendicular to the line $y = 4x - 3$.
23. For what value of the constant k is the line $x + y = k$ normal to the curve $y = x^2$?
24. For what value of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles. *Hint:* Where do the curves intersect? What are their slopes there?

Use a graphics utility to plot the following curves. Where does the curve have a horizontal tangent? Does the curve fail to have a tangent line anywhere?

25. $y = x^3(5 - x)^2$
26. $y = 2x^3 - 3x^2 - 12x + 1$
27. $y = |x^2 - 1| - x$
28. $y = |x + 1| - |x - 1|$
29. $y = (x^2 - 1)^{1/3}$
30. $y = ((x^2 - 1)^2)^{1/3}$
31. If line L is tangent to curve C at point P, then the smaller angle between L and the secant line PQ joining P to another point Q on C approaches 0 as Q approaches P along C. Is the converse true: if the angle between PQ and line L (which passes through P) approaches 0, must L be tangent to C?
32. Let $P(x)$ be a polynomial. If a is a real number, then $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

for some $n \ge 0$. If $\ell(x) = m(x-a) + b$, show that the straight line $y = \ell(x)$ is tangent to the graph of $y = P(x)$ at $x = a$ provided $P(x) - \ell(x) = (x-a)^2 Q(x)$, where $Q(x)$ is a polynomial.

2.2 The Derivative

A straight line has the property that its slope is the same at all points. For any other graph, however, the slope may vary from point to point. Thus the slope of the graph of $y = f(x)$ at the point x is itself a function of x. At any point x where the graph has a finite slope, we say that f is differentiable, and we call the slope the derivative of f. The derivative is therefore the limit of the Newton quotient.

DEFINITION 4

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

The domain of the derivative f' (read "f prime") is the set of numbers x in the domain of f where the graph of f has a *nonvertical* tangent line, and the value $f'(x_0)$ of f' at such a point x_0 is the slope of the tangent line to $y = f(x)$ there. Thus the equation of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is

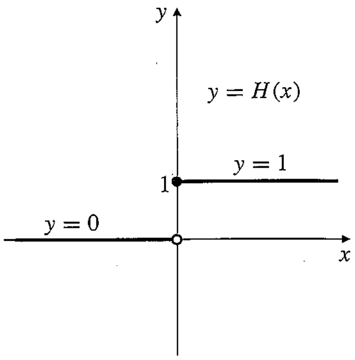

Figure 2.17 This function is not a derivative on $[-1, 1]$; it does not have the intermediate-value property.

are not. Although a derivative need not be a continuous function (see Exercise 18 in Section 2.6), it must, like a continuous function, have the intermediate-value property: on an interval $[a, b]$, a derivative $f'(x)$ takes on every value between $f'(a)$ and $f'(b)$. (See Exercise 19 in Section 2.6 for a proof of this fact.) An everywhere-defined step function such as the Heaviside $H(x)$ function considered in Example 1 in Section 1.4 does not have this property on, say, the interval $[-1, 1]$, so cannot be the derivative of a function on that interval. This argument does not apply to the signum function, which is the derivative of the absolute value function on any interval (see Example 4), even though it does not have the intermediate-value property on an interval containing the origin. Note, however, that the signum function is *itself* not defined at the origin.

If $g(x)$ is continuous on an interval I, then $g(x) = f'(x)$ for some function f that is differentiable on I. We will discuss this fact further in Chapter 5 and prove it in Appendix IV.

EXERCISES 2.2

Make rough sketches of the graphs of the derivatives of the functions in Exercises 1–4.

1. The function f graphed in Figure 2.18(a).

2. The function g graphed in Figure 2.18(b).

3. The function h graphed in Figure 2.18(c).

4. The function k graphed in Figure 2.18(d).

5. Where is the function f graphed in Figure 2.18(a) differentiable?

6. Where is the function g graphed in Figure 2.18(b) differentiable?

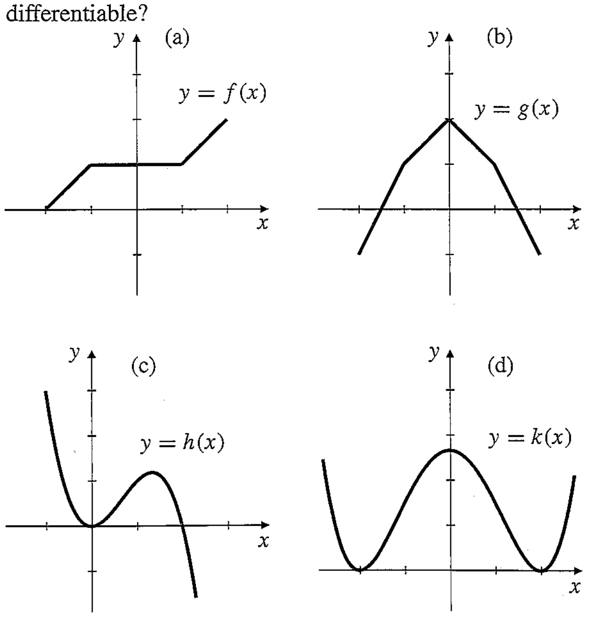

Figure 2.18

Use a graphics utility with differentiation capabilities to plot the graphs of the following functions and their derivatives. Observe the relationships between the graph of y and that of y' in each case. What features of the graph of y can you infer from the graph of y'?

7. $y = 3x - x^2 - 1$

8. $y = x^3 - 3x^2 + 2x + 1$

9. $y = |x^3 - x|$

10. $y = |x^2 - 1| - |x^2 - 4|$

In Exercises 11–24, (a) calculate the derivative of the given function directly from the definition of derivative, and (b) express the result of (a) using differentials.

11. $y = x^2 - 3x$

12. $f(x) = 1 + 4x - 5x^2$

13. $f(x) = x^3$

14. $s = \dfrac{1}{3 + 4t}$

15. $g(x) = \dfrac{2 - x}{2 + x}$

16. $y = \dfrac{1}{3}x^3 - x$

17. $F(t) = \sqrt{2t + 1}$

18. $f(x) = \dfrac{3}{4}\sqrt{2 - x}$

19. $y = x + \dfrac{1}{x}$

20. $z = \dfrac{s}{1 + s}$

21. $F(x) = \dfrac{1}{\sqrt{1 + x^2}}$

22. $y = \dfrac{1}{x^2}$

23. $y = \dfrac{1}{\sqrt{1 + x}}$

24. $f(t) = \dfrac{t^2 - 3}{t^2 + 3}$

25. How should the function $f(x) = x \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

26. How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

27. Where does $h(x) = |x^2 + 3x + 2|$ fail to be differentiable?

28. Using a calculator, find the slope of the secant line to $y = x^3 - 2x$ passing through the points corresponding to $x = 1$ and $x = 1 + \Delta x$, for several values of Δx of decreasing size, say $\Delta x = \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$. (Make a table.) Also, calculate $\dfrac{d}{dx}\left(x^3 - 2x\right)\Big|_{x=1}$ using the definition of derivative.

29. Repeat Exercise 28 for the function $f(x) = \dfrac{1}{x}$ and the points $x = 2$ and $x = 2 + \Delta x$.

Using the definition of derivative, find equations for the tangent lines to the curves in Exercises 30–33 at the points indicated.

30. $y = 5 + 4x - x^2$ at the point where $x = 2$

31. $y = \sqrt{x + 6}$ at the point $(3, 3)$

32. $y = \dfrac{t}{t^2 - 2}$ at the point where $t = -2$

33. $y = \dfrac{2}{t^2+t}$ at the point where $t = a$

Calculate the derivatives of the functions in Exercises 34–39 using the General Power Rule. Where is each derivative valid?

34. $f(x) = x^{-17}$

35. $g(t) = t^{22}$

36. $y = x^{1/3}$

37. $y = x^{-1/3}$

38. $t^{-2.25}$

39. $s^{119/4}$

In Exercises 40–50, you may use the formulas for derivatives established in this section.

40. Calculate $\left.\dfrac{d}{ds}\sqrt{s}\right|_{s=9}$.

41. Find $F'(\frac{1}{4})$ if $F(x) = \dfrac{1}{x}$.

42. Find $f'(8)$ if $f(x) = x^{-2/3}$.

43. Find $dy/dt\big|_{t=4}$ if $y = t^{1/4}$.

44. Find an equation of the straight line tangent to the curve $y = \sqrt{x}$ at $x = x_0$.

45. Find an equation of the straight line normal to the curve $y = 1/x$ at the point where $x = a$.

46. Show that the curve $y = x^2$ and the straight line $x + 4y = 18$ intersect at right angles at one of their two intersection points. *Hint:* Find the product of their slopes at their intersection points.

47. There are two distinct straight lines that pass through the point $(1, -3)$ and are tangent to the curve $y = x^2$. Find their equations. *Hint:* Draw a sketch. The points of tangency are not given; let them be denoted (a, a^2).

48. Find equations of two straight lines that have slope -2 and are tangent to the graph of $y = 1/x$.

49. Find the slope of a straight line that passes through the point $(-2, 0)$ and is tangent to the curve $y = \sqrt{x}$.

50. Show that there are two distinct tangent lines to the curve $y = x^2$ passing through the point (a, b) provided $b < a^2$. How many tangent lines to $y = x^2$ pass through (a, b) if $b = a^2$? if $b > a^2$?

51. Show that the derivative of an odd differentiable function is even and that the derivative of an even differentiable function is odd.

52. Prove the case $r = -n$ (n is a positive integer) of the General Power Rule; that is, prove that

$$\frac{d}{dx}x^{-n} = -n\,x^{-n-1}.$$

Use the factorization of a difference of nth powers given in this section.

53. Use the factoring of a difference of cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

to help you calculate the derivative of $f(x) = x^{1/3}$ directly from the definition of derivative.

54. Prove the General Power Rule for $\frac{d}{dx}x^r$, where $r = 1/n$, n being a positive integer. (*Hint:*

$$\frac{d}{dx}x^{1/n} = \lim_{h\to 0}\frac{(x+h)^{1/n} - x^{1/n}}{h} = \lim_{h\to 0}\frac{(x+h)^{1/n} - x^{1/n}}{((x+h)^{1/n})^n - (x^{1/n})^n}.$$

Apply the factorization of the difference of nth powers to the denominator of the latter quotient.)

55. Give a proof of the power rule $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers n using the Binomial Theorem:

$$(x+h)^n = x^n + \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1\times 2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^3 + \cdots + h^n.$$

56. Use right and left derivatives, $f'_+(a)$ and $f'_-(a)$, to define the concept of a half-line starting at $(a, f(a))$ being a right or left tangent to the graph of f at $x = a$. Show that the graph has a tangent line at $x = a$ if and only if it has right and left tangents that are opposite halves of the same straight line. What are the left and right tangents to the graphs of $y = x^{1/3}$, $y = x^{2/3}$, and $y = |x|$ at $x = 0$?

2.3 Differentiation Rules

If every derivative had to be calculated directly from the definition of derivative as in the examples of Section 2.2, calculus would indeed be a painful subject. Fortunately, there is an easier way. We will develop several general *differentiation rules* that enable us to calculate the derivatives of complicated combinations of functions easily if we already know the derivatives of the elementary functions from which they are constructed. For instance, we will be able to find the derivative of $\dfrac{x^2}{\sqrt{x^2+1}}$ if we know the derivatives of x^2 and $\sqrt{x}$. The rules we develop in this section tell us how to differentiate sums, constant multiples, products, and quotients of functions whose derivatives we already know. In Section 2.4 we will learn how to differentiate composite functions.

Before developing these differentiation rules we need to establish one obvious

EXERCISES 2.3

In Exercises 1–32, calculate the derivatives of the given functions. Simplify your answers whenever possible.

1. $y = 3x^2 - 5x - 7$
2. $y = 4x^{1/2} - \dfrac{5}{x}$
3. $f(x) = Ax^2 + Bx + C$
4. $f(x) = \dfrac{6}{x^3} + \dfrac{2}{x^2} - 2$
5. $z = \dfrac{s^5 - s^3}{15}$
6. $y = x^{45} - x^{-45}$
7. $g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$
8. $y = 3\sqrt[3]{t^2} - \dfrac{2}{\sqrt{t^3}}$
9. $u = \dfrac{3}{5}x^{5/3} - \dfrac{5}{3}x^{-3/5}$
10. $F(x) = (3x - 2)(1 - 5x)$
11. $y = \sqrt{x}\left(5 - x - \dfrac{x^2}{3}\right)$
12. $g(t) = \dfrac{1}{2t - 3}$
13. $y = \dfrac{1}{x^2 + 5x}$
14. $y = \dfrac{4}{3 - x}$
15. $f(t) = \dfrac{\pi}{2 - \pi t}$
16. $g(y) = \dfrac{2}{1 - y^2}$
17. $f(x) = \dfrac{1 - 4x^2}{x^3}$
18. $g(u) = \dfrac{u\sqrt{u} - 3}{u^2}$
19. $y = \dfrac{2 + t + t^2}{\sqrt{t}}$
20. $z = \dfrac{x - 1}{x^{2/3}}$
21. $f(x) = \dfrac{3 - 4x}{3 + 4x}$
22. $z = \dfrac{t^2 + 2t}{t^2 - 1}$
23. $s = \dfrac{1 + \sqrt{t}}{1 - \sqrt{t}}$
24. $f(x) = \dfrac{x^3 - 4}{x + 1}$
25. $f(x) = \dfrac{ax + b}{cx + d}$
26. $F(t) = \dfrac{t^2 + 7t - 8}{t^2 - t + 1}$
27. $f(x) = (1 + x)(1 + 2x)(1 + 3x)(1 + 4x)$
28. $f(r) = (r^{-2} + r^{-3} - 4)(r^2 + r^3 + 1)$
29. $y = (x^2 + 4)(\sqrt{x} + 1)(5x^{2/3} - 2)$
30. $y = \dfrac{(x^2 + 1)(x^3 + 2)}{(x^2 + 2)(x^3 + 1)}$
31. $y = \dfrac{x}{2x + \dfrac{1}{3x + 1}}$
32. $f(x) = \dfrac{(\sqrt{x} - 1)(2 - x)(1 - x^2)}{\sqrt{x}(3 + 2x)}$

Calculate the derivatives in Exercises 33–36, given that $f(2) = 2$ and $f'(2) = 3$.

33. $\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\Bigg|_{x=2}$
34. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2}\right)\Bigg|_{x=2}$
35. $\dfrac{d}{dx}\left(x^2 f(x)\right)\Big|_{x=2}$
36. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2 + f(x)}\right)\Bigg|_{x=2}$
37. Find $\dfrac{d}{dx}\left(\dfrac{x^2 - 4}{x^2 + 4}\right)\Bigg|_{x=-2}$.
38. Find $\dfrac{d}{dt}\left(\dfrac{t(1 + \sqrt{t})}{5 - t}\right)\Bigg|_{t=4}$.
39. If $f(x) = \dfrac{\sqrt{x}}{x + 1}$, find $f'(2)$.
40. Find $\dfrac{d}{dt}\Big((1 + t)(1 + 2t)(1 + 3t)(1 + 4t)\Big)\Big|_{t=0}$.
41. Find an equation of the tangent line to $y = \dfrac{2}{3 - 4\sqrt{x}}$ at the point $(1, -2)$.
42. Find equations of the tangent and normal to $y = \dfrac{x + 1}{x - 1}$ at $x = 2$.
43. Find the points on the curve $y = x + 1/x$ where the tangent line is horizontal.
44. Find the equations of all horizontal lines that are tangent to the curve $y = x^2(4 - x^2)$.
45. Find the coordinates of all points where the curve $y = \dfrac{1}{x^2 + x + 1}$ has a horizontal tangent line.
46. Find the coordinates of points on the curve $y = \dfrac{x + 1}{x + 2}$ where the tangent line is parallel to the line $y = 4x$.
47. Find the equation of the straight line that passes through the point $(0, b)$ and is tangent to the curve $y = 1/x$. Assume $b \neq 0$.
48. Show that the curve $y = x^2$ intersects the curve $y = 1/\sqrt{x}$ at right angles.
49. Find two straight lines that are tangent to $y = x^3$ and pass through the point $(2, 8)$.
50. Find two straight lines that are tangent to $y = x^2/(x - 1)$ and pass through the point $(2, 0)$.
51. **(A Square Root Rule)** Show that if f is differentiable at x and $f(x) > 0$, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}.$$

Use this Square Root Rule to find the derivative of $\sqrt{x^2 + 1}$.

52. Show that $f(x) = |x^3|$ is differentiable at every real number x, and find its derivative.

Mathematical Induction

53. Use mathematical induction to prove that $\dfrac{d}{dx}x^{n/2} = \dfrac{n}{2}x^{(n/2)-1}$ for every positive integer n. Then use the Reciprocal Rule to get the same result for negative integers n.
54. Use mathematical induction to prove the formula for the derivative of a product of n functions given earlier in this section.

EXERCISES 2.4

Find the derivatives of the functions in Exercises 1–16.

1. $y = (2x+3)^6$

2. $y = \left(1 - \frac{x}{3}\right)^{99}$

3. $f(x) = (4 - x^2)^{10}$

4. $y = \sqrt{1 - 3x^2}$

5. $F(t) = \left(2 + \frac{3}{t}\right)^{-10}$

6. $(1 + x^{2/3})^{3/2}$

7. $\frac{3}{5 - 4x}$

8. $(1 - 2t^2)^{-3/2}$

9. $y = |1 - x^2|$

10. $f(t) = |2 + t^3|$

11. $y = 4x + |4x - 1|$

12. $y = (2 + |x|^3)^{1/3}$

13. $y = \frac{1}{2 + \sqrt{3x+4}}$

14. $f(x) = \left(1 + \sqrt{\frac{x-2}{3}}\right)^4$

15. $z = \left(u + \frac{1}{u-1}\right)^{-5/3}$

16. $y = \frac{x^5\sqrt{3+x^6}}{(4+x^2)^3}$

17. Sketch the graph of the function in Exercise 10.

18. Sketch the graph of the function in Exercise 11.

Verify that the General Power Rule holds for the functions in Exercises 19–21.

19. $x^{1/4} = \sqrt{\sqrt{x}}$

20. $x^{3/4} = \sqrt{x\sqrt{x}}$

21. $x^{3/2} = \sqrt{(x^3)}$

In Exercises 22–29, express the derivative of the given function in terms of the derivative f' of the differentiable function f.

22. $f(2t+3)$

23. $f(5x - x^2)$

24. $\left[f\left(\frac{2}{x}\right)\right]^3$

25. $\sqrt{3 + 2f(x)}$

26. $f\left(\sqrt{3+2t}\right)$

27. $f\left(3 + 2\sqrt{x}\right)$

28. $f\Big(2f(3f(x))\Big)$

29. $f\Big(2 - 3f(4 - 5t)\Big)$

30. Find $\left.\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\right|_{x=-2}$.

31. Find $\left.\frac{d}{dt}\sqrt{3t-7}\right|_{t=3}$.

32. If $f(x) = \frac{1}{\sqrt{2x+1}}$, find $f'(4)$.

33. If $y = (x^3+9)^{17/2}$, find $\left.y'\right|_{x=-2}$.

34. Find $F'(0)$ if $F(x) = (1+x)(2+x)^2(3+x)^3(4+x)^4$.

35. Calculate y' if $y = (x + ((3x)^5 - 2)^{-1/2})^{-6}$. Try to do it all in one step.

In Exercises 36–39, find an equation of the tangent line to the given curve at the given point.

36. $y = \sqrt{1+2x^2}$ at $x = 2$

37. $y = (1 + x^{2/3})^{3/2}$ at $x = -1$

38. $y = (ax+b)^8$ at $x = b/a$

39. $y = 1/(x^2 - x + 3)^{3/2}$ at $x = -2$

40. Show that the derivative of $f(x) = (x-a)^m(x-b)^n$ vanishes at some point between a and b if m and n are positive integers.

Use Maple or another computer algebra system to evaluate and simplify the derivatives of the functions in Exercises 41–44.

41. $y = \sqrt{x^2+1} + \frac{1}{(x^2+1)^{3/2}}$

42. $y = \frac{(x^2-1)(x^2-4)(x^2-9)}{x^6}$

43. $\left.\frac{dy}{dt}\right|_{t=2}$ if $y = (t+1)(t^2+2)(t^3+3)(t^4+4)(t^5+5)$

44. $f'(1)$ if $f(x) = \frac{(x^2+3)^{1/2}(x^3+7)^{1/3}}{(x^4+15)^{1/4}}$

45. Does the Chain Rule enable you to calculate the derivatives of $|x|^2$ and $|x^2|$ at $x = 0$? Do these functions have derivatives at $x = 0$? Why?

46. What is wrong with the following "proof" of the Chain Rule? Let $k = g(x+h) - g(x)$. Then $\lim_{h\to 0} k = 0$. Thus

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\,\frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(g(x)+k) - f(g(x))}{k}\,\frac{g(x+h) - g(x)}{h} \\
&= f'(g(x))\,g'(x).
\end{aligned}$$

2.5 Derivatives of Trigonometric Functions

The trigonometric functions, especially sine and cosine, play a very important role in the mathematical modelling of real-world phenomena. In particular, they arise whenever quantities fluctuate in a periodic way. Elastic motions, vibrations, and waves of all kinds naturally involve the trigonometric functions, and many physical and mechanical laws are formulated as differential equations having these functions as solutions.

In this section we will calculate the derivatives of the six trigonometric functions. We only have to work hard for one of them, sine; the others then follow from known identities and the differentiation rules of Section 2.3.

EXAMPLE 4 Verify the derivative formulas for $\tan x$ and $\sec x$.

Solution We use the Quotient Rule for tangent and the Reciprocal Rule for secant:

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \\ \frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{-1}{\cos^2 x}\frac{d}{dx}(\cos x) \\ &= \frac{-1}{\cos^2 x}(-\sin x) = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \\ &= \sec x\ \tan x.\end{aligned}$$

EXAMPLE 5 (a) $\frac{d}{dx}\left[3x + \cot\left(\frac{x}{2}\right)\right] = 3 + \left[-\csc^2\left(\frac{x}{2}\right)\right]\frac{1}{2} = 3 - \frac{1}{2}\csc^2\left(\frac{x}{2}\right)$

(b) $$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sin(2x)}\right) &= \frac{d}{dx}(3\csc(2x)) \\ &= 3(-\csc(2x)\cot(2x))(2) = -6\csc(2x)\cot(2x).\end{aligned}$$

EXAMPLE 6 Find the tangent and normal lines to the curve $y = \tan(\pi x/4)$ at the point $(1, 1)$.

Solution The slope of the tangent to $y = \tan(\pi x/4)$ at $(1, 1)$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{4}\sec^2(\pi x/4)\Big|_{x=1} = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\left(\sqrt{2}\right)^2 = \frac{\pi}{2}.$$

The tangent is the line

$$y = 1 + \frac{\pi}{2}(x-1), \qquad \text{or} \qquad y = \frac{\pi x}{2} - \frac{\pi}{2} + 1.$$

The normal has slope $m = -2/\pi$, so its point-slope equation is

$$y = 1 - \frac{2}{\pi}(x-1), \qquad \text{or} \qquad y = -\frac{2x}{\pi} + \frac{2}{\pi} + 1.$$

EXERCISES 2.5

1. Verify the formula for the derivative of $\csc x = 1/(\sin x)$.
2. Verify the formula for the derivative of $\cot x = (\cos x)/(\sin x)$.

Find the derivatives of the functions in Exercises 3–36. Simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.

3. $y = \cos 3x$
4. $y = \sin \frac{x}{5}$
5. $y = \tan \pi x$
6. $y = \sec ax$
7. $y = \cot(4 - 3x)$
8. $y = \sin((\pi - x)/3)$
9. $f(x) = \cos(s - rx)$
10. $y = \sin(Ax + B)$
11. $\sin(\pi x^2)$
12. $\cos(\sqrt{x})$
13. $y = \sqrt{1 + \cos x}$
14. $\sin(2\cos x)$
15. $f(x) = \cos(x + \sin x)$
16. $g(\theta) = \tan(\theta \sin\theta)$
17. $u = \sin^3(\pi x/2)$
18. $y = \sec(1/x)$

19. $F(t) = \sin at \cos at$

20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$

21. $\sin(2x) - \cos(2x)$

22. $\cos^2 x - \sin^2 x$

23. $\tan x + \cot x$

24. $\sec x - \csc x$

25. $\tan x - x$

26. $\tan(3x)\cot(3x)$

27. $t\cos t - \sin t$

28. $t\sin t + \cos t$

29. $\dfrac{\sin x}{1+\cos x}$

30. $\dfrac{\cos x}{1+\sin x}$

31. $x^2\cos(3x)$

32. $g(t) = \sqrt{(\sin t)/t}$

33. $v = \sec(x^2)\tan(x^2)$

34. $z = \dfrac{\sin\sqrt{x}}{1+\cos\sqrt{x}}$

35. $\sin(\cos(\tan t))$

36. $f(s) = \cos(s + \cos(s + \cos s))$

37. Given that $\sin 2x = 2\sin x\cos x$, deduce that $\cos 2x = \cos^2 x - \sin^2 x$.

38. Given that $\cos 2x = \cos^2 x - \sin^2 x$, deduce that $\sin 2x = 2\sin x\cos x$.

In Exercises 39–42, find equations for the lines that are tangent and normal to the curve $y = f(x)$ at the given point.

39. $y = \sin x,\ (\pi, 0)$

40. $y = \tan(2x),\ (0, 0)$

41. $y = \sqrt{2}\cos(x/4),\ (\pi, 1)$

42. $y = \cos^2 x,\ \left(\dfrac{\pi}{3}, \dfrac{1}{4}\right)$

43. Find an equation of the line tangent to the curve $y = \sin(x°)$ at the point where $x = 45$.

44. Find an equation of the straight line normal to $y = \sec(x°)$ at the point where $x = 60$.

45. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.

46. Find the points on the curve $y = \tan(2x)$, $-\pi/4 < x < \pi/4$, where the normal is parallel to the line $y = -x/8$.

47. Show that the graphs of $y = \sin x$, $y = \cos x$, $y = \sec x$, and $y = \csc x$ have horizontal tangents.

48. Show that the graphs of $y = \tan x$ and $y = \cot x$ never have horizontal tangents.

Do the graphs of the functions in Exercises 49–52 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not?

49. $y = x + \sin x$

50. $y = 2x + \sin x$

51. $y = x + 2\sin x$

52. $y = x + 2\cos x$

Find the limits in Exercises 53–56.

53. $\displaystyle\lim_{x\to 0}\frac{\tan(2x)}{x}$

54. $\displaystyle\lim_{x\to\pi}\sec(1+\cos x)$

55. $\displaystyle\lim_{x\to 0}(x^2\csc x\cot x)$

56. $\displaystyle\lim_{x\to 0}\cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right)$

57. Use the method of Example 1 to evaluate $\displaystyle\lim_{h\to 0}\frac{1-\cos h}{h^2}$.

58. Find values of a and b that make

$$f(x) = \begin{cases} ax + b, & x < 0 \\ 2\sin x + 3\cos x, & x \ge 0 \end{cases}$$

differentiable at $x = 0$.

59. How many straight lines that pass through the origin are tangent to $y = \cos x$? Find (to 6 decimal places) the slopes of the two such lines that have the largest positive slopes.

Use Maple or another computer algebra system to evaluate and simplify the derivatives of the functions in Exercises 60–61.

60. $\displaystyle\frac{d}{dx}\frac{x\cos(x\sin x)}{x+\cos(x\cos x)}\bigg|_{x=0}$

61. $\displaystyle\frac{d}{dx}\left(\sqrt{2x^2+3}\sin(x^2) - \frac{(2x^2+3)^{3/2}\cos(x^2)}{x}\right)\bigg|_{x=\sqrt{\pi}}$

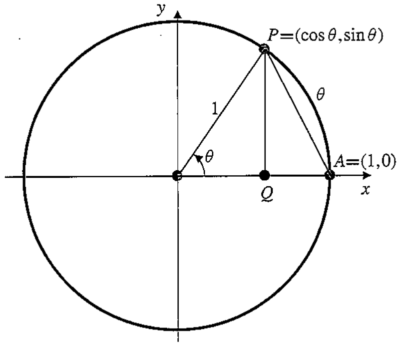

Figure 2.24

62. **(The continuity of sine and cosine)**

(a) Prove that

$$\lim_{\theta\to 0}\sin\theta = 0 \quad\text{and}\quad \lim_{\theta\to 0}\cos\theta = 1$$

as follows: Use the fact that the length of chord AP is less than the length of arc AP in Figure 2.24 to show that

$$\sin^2\theta + (1-\cos\theta)^2 < \theta^2.$$

Then deduce that $0 \le |\sin\theta| < |\theta|$ and $0 \le |1-\cos\theta| < |\theta|$. Then use the Squeeze Theorem from Section 1.2.

(b) Part (a) says that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$. Use the addition formulas to prove that they are therefore continuous at every θ.

2.6 Higher-Order Derivatives

If the derivative $y' = f'(x)$ of a function $y = f(x)$ is itself differentiable at x, we can calculate *its* derivative, which we call the **second derivative** of f and denote by

```
> f := x -> x^5; fpp := D(D(f)); (D@@3)(f)(a);
```

$$f := x \to x^5$$
$$fpp := x \to 20\,x^3$$
$$60\,a^2$$

EXERCISES 2.6

Find y', y'', and y''' for the functions in Exercises 1–12.

1. $y = (3 - 2x)^7$
2. $y = x^2 - \dfrac{1}{x}$
3. $y = \dfrac{6}{(x-1)^2}$
4. $y = \sqrt{ax+b}$
5. $y = x^{1/3} - x^{-1/3}$
6. $y = x^{10} + 2x^8$
7. $y = (x^2+3)\sqrt{x}$
8. $y = \dfrac{x-1}{x+1}$
9. $y = \tan x$
10. $y = \sec x$
11. $y = \cos(x^2)$
12. $y = \dfrac{\sin x}{x}$

In Exercises 13–23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

13. $f(x) = \dfrac{1}{x}$
14. $f(x) = \dfrac{1}{x^2}$
15. $f(x) = \dfrac{1}{2-x}$
16. $f(x) = \sqrt{x}$
17. $f(x) = \dfrac{1}{a+bx}$
18. $f(x) = x^{2/3}$
19. $f(x) = \cos(ax)$
20. $f(x) = x\cos x$
21. $f(x) = x\sin(ax)$
22. $f(x) = \dfrac{1}{|x|}$
23. $f(x) = \sqrt{1-3x}$
24. If $y = \tan kx$, show that $y'' = 2k^2y(1+y^2)$.
25. If $y = \sec kx$, show that $y'' = k^2y(2y^2 - 1)$.
26. Use mathematical induction to prove that the nth derivative of $y = \sin(ax+b)$ is given by the formula asserted at the end of Example 5.
27. Use mathematical induction to prove that the nth derivative of $y = \tan x$ is of the form $P_{n+1}(\tan x)$, where P_{n+1} is a polynomial of degree $n+1$.
28. If f and g are twice-differentiable functions, show that $(fg)'' = f''g + 2f'g' + fg''$.
29. State and prove the results analogous to that of Exercise 28 but for $(fg)^{(3)}$ and $(fg)^{(4)}$. Can you guess the formula for $(fg)^{(n)}$?
30. If $f''(x)$ exists on an interval I and if f vanishes at at least three distinct points of I, prove that f'' must vanish at some point in I.
31. Generalize Exercise 30 to a function for which $f^{(n)}$ exists on I and for which f vanishes at at least $n+1$ distinct points in I.
32. Suppose f is twice differentiable on an interval I (i.e., f'' exists on I). Suppose that the points 0 and 2 belong to I and that $f(0) = f(1) = 0$ and $f(2) = 1$. Prove that:
    (a) $f'(a) = \dfrac{1}{2}$ for some point a in I.
    (b) $f''(b) > \dfrac{1}{2}$ for some point b in I.
    (c) $f'(c) = \dfrac{1}{7}$ for some point c in I.

2.7 Using Differentials and Derivatives

In this section we will look at some examples of ways in which derivatives are used to represent and interpret changes and rates of change in the world around us. It is natural to think of change in terms of dependence on time, such as the velocity of a moving object, but there is no need to be so restrictive. Change with respect to variables other than time can be treated in the same way. For example, a physician may want to know how small changes in dosage can affect the body's response to a drug. An economist may want to study how foreign investment changes with respect to variations in a country's interest rates. These questions can all be formulated in terms of rate of change of a function with respect to a variable.

Approximating Small Changes

If one quantity, say y, is a function of another quantity x, that is,

$$y = f(x),$$

PROOF Note that $g(b) \neq g(a)$; otherwise, there would be some number in (a, b) where $g' = 0$. Hence, neither denominator above can be zero. Apply the Mean-Value Theorem to

$$h(x) = \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(x) - f(a)\big).$$

Since $h(a) = h(b) = 0$, there exists c in (a, b) such that $h'(c) = 0$. Thus,

$$\big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0,$$

and the result follows on division by the g factors. ∎

EXERCISES 2.8

In Exercises 1–3, illustrate the Mean-Value Theorem by finding any points in the open interval (a, b) where the tangent line to $y = f(x)$ is parallel to the chord line joining $(a, f(a))$ and $(b, f(b))$.

1. $f(x) = x^2$ on $[a, b]$ **2.** $f(x) = \dfrac{1}{x}$ on $[1, 2]$

3. $f(x) = x^3 - 3x + 1$ on $[-2, 2]$

4. By applying the Mean-Value Theorem to $f(x) = \cos x + \dfrac{x^2}{2}$ on the interval $[0, x]$, and using the result of Example 2, show that

$$\cos x > 1 - \frac{x^2}{2}$$

for $x > 0$. This inequality is also true for $x < 0$. Why?

5. Show that $\tan x > x$ for $0 < x < \pi/2$.

6. Let $r > 1$. If $x > 0$ or $-1 \le x < 0$, show that $(1 + x)^r > 1 + rx$.

7. Let $0 < r < 1$. If $x > 0$ or $-1 \le x < 0$, show that $(1 + x)^r < 1 + rx$.

Find the intervals of increase and decrease of the functions in Exercises 8–15.

8. $f(x) = x^2 + 2x + 2$ **9.** $f(x) = x^3 - 4x + 1$

10. $f(x) = x^3 + 4x + 1$ **11.** $f(x) = (x^2 - 4)^2$

12. $f(x) = \dfrac{1}{x^2 + 1}$ **13.** $f(x) = x^3(5 - x)^2$

14. $f(x) = x - 2\sin x$ **15.** $f(x) = x + \sin x$

16. If $f(x)$ is differentiable on an interval I and vanishes at $n \ge 2$ distinct points of I, prove that $f'(x)$ must vanish at at least $n - 1$ points in I.

17. What is wrong with the following "proof" of the Generalized Mean-Value Theorem? By the Mean-Value Theorem, $f(b) - f(a) = (b - a)f'(c)$ for some c between a and b and, similarly, $g(b) - g(a) = (b - a)g'(c)$ for some such c. Hence, $(f(b) - f(a))/(g(b) - g(a)) = f'(c)/g'(c)$, as required.

18. Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists at every x but f' is not continuous at $x = 0$. This proves the assertion (made at the end of Section 2.2) that a derivative, defined on an interval, need not be continuous there.

19. Prove the assertion (made at the end of Section 2.2) that a derivative, defined on an interval, must have the intermediate-value property. (*Hint:* Assume that f' exists on $[a, b]$ and $f'(a) \neq f'(b)$. If k lies between $f'(a)$ and $f'(b)$, show that the function g defined by $g(x) = f(x) - kx$ must have *either* a maximum value *or* a minimum value on $[a, b]$ occurring at an interior point c in (a, b). Deduce that $f'(c) = k$.)

20. Let $f(x) = \begin{cases} x + 2x^2\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) Show that $f'(0) = 1$. (*Hint:* Use the definition of derivative.)

(b) Show that any interval containing $x = 0$ also contains points where $f'(x) < 0$, so f cannot be increasing on such an interval.

2.9 Implicit Differentiation

We know how to find the slope of a curve that is the graph of a function $y = f(x)$ by calculating the derivative of f. But not all curves are the graphs of such functions. To be the graph of a function $f(x)$, the curve must not intersect any vertical lines at more than one point.

$$\frac{d}{dx}x^r = r\,x^{r-1}$$

for integer exponents r and a few special rational exponents such as $r = 1/2$. Using implicit differentiation, we can give the proof for any rational exponent $r = m/n$, where m and n are integers, and $n \neq 0$.

If $y = x^{m/n}$, then $y^n = x^m$. Differentiating implicitly with respect to x, we obtain

$$n\,y^{n-1}\frac{dy}{dx} = m\,x^{m-1}, \qquad \text{so}$$

$$\frac{dy}{dx} = \frac{m}{n}x^{m-1}y^{1-n} = \frac{m}{n}x^{m-1}x^{(m/n)(1-n)} = \frac{m}{n}x^{m-1+(m/n)-m} = \frac{m}{n}x^{(m/n)-1}.$$

EXERCISES 2.9

In Exercises 1–8, find dy/dx in terms of x and y.

1. $xy - x + 2y = 1$

2. $x^3 + y^3 = 1$

3. $x^2 + xy = y^3$

4. $x^3y + xy^5 = 2$

5. $x^2y^3 = 2x - y$

6. $x^2 + 4(y-1)^2 = 4$

7. $\dfrac{x-y}{x+y} = \dfrac{x^2}{y} + 1$

8. $x\sqrt{x+y} = 8 - xy$

In Exercises 9–16, find an equation of the tangent to the given curve at the given point.

9. $2x^2 + 3y^2 = 5$ at $(1, 1)$

10. $x^2y^3 - x^3y^2 = 12$ at $(-1, 2)$

11. $\dfrac{x}{y} + \left(\dfrac{y}{x}\right)^3 = 2$ at $(-1, -1)$

12. $x + 2y + 1 = \dfrac{y^2}{x-1}$ at $(2, -1)$

13. $2x + y - \sqrt{2}\sin(xy) = \pi/2$ at $\left(\dfrac{\pi}{4}, 1\right)$

14. $\tan(xy^2) = \dfrac{2xy}{\pi}$ at $\left(-\pi, \dfrac{1}{2}\right)$

15. $x\sin(xy - y^2) = x^2 - 1$ at $(1, 1)$

16. $\cos\left(\dfrac{\pi y}{x}\right) = \dfrac{x^2}{y} - \dfrac{17}{2}$ at $(3, 1)$

In Exercises 17–20, find y'' in terms of x and y.

17. $xy = x + y$

18. $x^2 + 4y^2 = 4$

19. $x^3 - y^2 + y^3 = x$

20. $x^3 - 3xy + y^3 = 1$

21. For $x^2 + y^2 = a^2$ show that $y'' = -\dfrac{a^2}{y^3}$.

22. For $Ax^2 + By^2 = C$ show that $y'' = -\dfrac{AC}{B^2y^3}$.

Use Maple or another computer algebra program to find the values requested in Exercises 23–26.

23. Find the slope of $x + y^2 + y\sin x = y^3 + \pi$ at $(\pi, 1)$.

24. Find the slope of $\dfrac{x+\sqrt{y}}{y+\sqrt{x}} = \dfrac{3y-9x}{x+y}$ at the point $(1, 4)$.

25. If $x + y^5 + 1 = y + x^4 + xy^2$, find d^2y/dx^2 at $(1, 1)$.

26. If $x^3y + xy^3 = 11$, find d^3y/dx^3 at $(1, 2)$.

27. Show that the ellipse $x^2 + 2y^2 = 2$ and the hyperbola $2x^2 - 2y^2 = 1$ intersect at right angles.

28. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ intersect at right angles if $A^2 \le a^2$ and $a^2 - b^2 = A^2 + B^2$. (This says that the ellipse and the hyperbola have the same foci.)

29. If $z = \tan\dfrac{x}{2}$, show that

$$\frac{dx}{dz} = \frac{2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}, \quad \text{and } \cos x = \frac{1-z^2}{1+z^2}.$$

30. Use implicit differentiation to find y' if $(x-y)/(x+y) = x/y + 1$. Now show that there are, in fact, no points on that curve, so the derivative you calculated is meaningless. This is another example that demonstrates the dangers of calculating something when you don't know whether or not it exists.

2.10 Antiderivatives and Initial-Value Problems

Throughout this chapter we have been concerned with the problem of finding the derivative f' of a given function f. The reverse problem—given the derivative f', find f—is also interesting and important. It is the problem studied in *integral calculus* and is generally more difficult to solve than the problem of finding a derivative. We will take a preliminary look at this problem in this section and will return to it in more detail in Chapter 5.

EXERCISES 2.10

In Exercises 1–14, find the given indefinite integrals.

1. $\int 5\,dx$

2. $\int x^2\,dx$

3. $\int \sqrt{x}\,dx$

4. $\int x^{12}\,dx$

5. $\int x^3\,dx$

6. $\int (x+\cos x)\,dx$

7. $\int \tan x \cos x\,dx$

8. $\int \frac{1+\cos^3 x}{\cos^2 x}\,dx$

9. $\int (a^2-x^2)\,dx$

10. $\int (A+Bx+Cx^2)\,dx$

11. $\int (2x^{1/2}+3x^{1/3})\,dx$

12. $\int \frac{6(x-1)}{x^{4/3}}\,dx$

13. $\int \left(\frac{x^3}{3}-\frac{x^2}{2}+x-1\right)\,dx$

14. $105\int (1+t^2+t^4+t^6)\,dt$

In Exercises 15–22, find the given indefinite integrals. This may require guessing the form of an antiderivative and then checking by differentiation. For instance, you might suspect that $\int \cos(5x-2)\,dx = k\sin(5x-2)+C$ for some k. Differentiating the answer shows that k must be $1/5$.

15. $\int \cos(2x)\,dx$

16. $\int \sin\left(\frac{x}{2}\right)\,dx$

17. $\int \frac{dx}{(1+x)^2}$

18. $\int \sec(1-x)\tan(1-x)\,dx$

19. $\int \sqrt{2x+3}\,dx$

20. $\int \frac{4}{\sqrt{x+1}}\,dx$

21. $\int 2x\sin(x^2)\,dx$

22. $\int \frac{2x}{\sqrt{x^2+1}}\,dx$

Use trigonometric identities such as $\sec^2 x = 1+\tan^2 x$, $\sin(2x) = 2\sin x\cos x$, and $\cos(2x) = 2\cos^2 x - 1 = 1-2\sin^2 x$ to help you evaluate the indefinite integrals in Exercises 23–26.

23. $\int \tan^2 x\,dx$

24. $\int \sin x\cos x\,dx$

25. $\int \cos^2 x\,dx$

26. $\int \sin^2 x\,dx$

Differential equations

In Exercises 27–42, find the solution $y = y(x)$ to the given initial-value problem. On what interval is the solution valid? (Note that exercises involving differential equations are prefixed with the symbol ✱.)

27. $\begin{cases} y' = x-2 \\ y(0) = 3 \end{cases}$

28. $\begin{cases} y' = x^{-2}-x^{-3} \\ y(-1) = 0 \end{cases}$

29. $\begin{cases} y' = 3\sqrt{x} \\ y(4) = 1 \end{cases}$

30. $\begin{cases} y' = x^{1/3} \\ y(0) = 5 \end{cases}$

31. $\begin{cases} y' = Ax^2+Bx+C \\ y(1) = 1 \end{cases}$

32. $\begin{cases} y' = x^{-9/7} \\ y(1) = -4 \end{cases}$

33. $\begin{cases} y' = \cos x \\ y(\pi/6) = 2 \end{cases}$

34. $\begin{cases} y' = \sin(2x) \\ y(\pi/2) = 1 \end{cases}$

35. $\begin{cases} y' = \sec^2 x \\ y(0) = 1 \end{cases}$

36. $\begin{cases} y' = \sec^2 x \\ y(\pi) = 1 \end{cases}$

37. $\begin{cases} y'' = 2 \\ y'(0) = 5 \\ y(0) = -3 \end{cases}$

38. $\begin{cases} y'' = x^{-4} \\ y'(1) = 2 \\ y(1) = 1 \end{cases}$

39. $\begin{cases} y'' = x^3-1 \\ y'(0) = 0 \\ y(0) = 8 \end{cases}$

40. $\begin{cases} y'' = 5x^2-3x^{-1/2} \\ y'(1) = 2 \\ y(1) = 0 \end{cases}$

41. $\begin{cases} y'' = \cos x \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

42. $\begin{cases} y'' = x+\sin x \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$

43. Show that for any constants A and B the function $y = y(x) = Ax + B/x$ satisfies the *second-order differential equation* $x^2y''+xy'-y = 0$ for $x \neq 0$. Find a function y satisfying the initial-value problem:

$$\begin{cases} x^2y''+xy'-y = 0 \qquad (x>0) \\ y(1) = 2 \\ y'(1) = 4. \end{cases}$$

44. Show that for any constants A and B the function $y = Ax^{r_1}+Bx^{r_2}$ satisfies, for $x>0$, the differential equation $ax^2y''+bxy'+cy = 0$, provided that r_1 and r_2 are two distinct rational roots of the quadratic equation $ar(r-1)+br+c = 0$.

Use the result of Exercise 44 to solve the initial-value problems in Exercises 45–46 on the interval $x>0$.

45. $\begin{cases} 4x^2y''+4xy'-y = 0 \\ y(4) = 2 \\ y'(4) = -2 \end{cases}$

46. $\begin{cases} x^2y''-6y = 0 \\ y(1) = 1 \\ y'(1) = 1 \end{cases}$

2.11 Velocity and Acceleration

Velocity and Speed

Suppose that an object is moving along a straight line (say the x-axis) so that its position

CHAPTER REVIEW

Key Ideas

- **What do the following statements and phrases mean?**
  - ◇ Line L is tangent to curve C at point P.
  - ◇ the Newton quotient of $f(x)$ at $x = a$
  - ◇ the derivative $f'(x)$ of the function $f(x)$
  - ◇ f is differentiable at $x = a$.
  - ◇ the slope of the graph $y = f(x)$ at $x = a$
  - ◇ f is increasing (or decreasing) on interval I.
  - ◇ f is nondecreasing (or nonincreasing) on interval I.
  - ◇ the average rate of change of $f(x)$ on $[a, b]$
  - ◇ the rate of change of $f(x)$ at $x = a$
  - ◇ c is a critical point of $f(x)$.
  - ◇ the second derivative of $f(x)$ at $x = a$
  - ◇ an antiderivative of f on interval I
  - ◇ the indefinite integral of f on interval I
  - ◇ differential equation ◇ initial-value problem
  - ◇ velocity ◇ speed ◇ acceleration
- **State the following differentiation rules:**
  - ◇ the rule for differentiating a sum of functions
  - ◇ the rule for differentiating a constant multiple of a function
  - ◇ the Product Rule ◇ the Reciprocal Rule
  - ◇ the Quotient Rule ◇ the Chain Rule
- **State the Mean-Value Theorem.**
- **State the Generalized Mean-Value Theorem.**
- **State the derivatives of the following functions:**
  - ◇ x ◇ x^2 ◇ $1/x$ ◇ $\sqrt{x}$
  - ◇ x^n ◇ $|x|$ ◇ $\sin x$ ◇ $\cos x$
  - ◇ $\tan x$ ◇ $\cot x$ ◇ $\sec x$ ◇ $\csc x$
- **What is a proof by mathematical induction?**

Review Exercises

Use the definition of derivative to calculate the derivatives in Exercises 1–4.

1. $\dfrac{dy}{dx}$ if $y = (3x + 1)^2$ **2.** $\dfrac{d}{dx}\sqrt{1 - x^2}$

3. $f'(2)$ if $f(x) = \dfrac{4}{x^2}$ **4.** $g'(9)$ if $g(t) = \dfrac{t - 5}{1 + \sqrt{t}}$

5. Find the tangent to $y = \cos(\pi x)$ at $x = 1/6$.

6. Find the normal to $y = \tan(x/4)$ at $x = \pi$.

Calculate the derivatives of the functions in Exercises 7–12.

7. $\dfrac{1}{x - \sin x}$ **8.** $\dfrac{1 + x + x^2 + x^3}{x^4}$

9. $(4 - x^{2/5})^{-5/2}$ **10.** $\sqrt{2 + \cos^2 x}$

11. $\tan\theta - \theta\sec^2\theta$ **12.** $\dfrac{\sqrt{1 + t^2} - 1}{\sqrt{1 + t^2} + 1}$

Evaluate the limits in Exercises 13–16 by interpreting each as a derivative.

13. $\displaystyle\lim_{h\to 0}\frac{(x + h)^{20} - x^{20}}{h}$ **14.** $\displaystyle\lim_{x\to 2}\frac{\sqrt{4x + 1} - 3}{x - 2}$

15. $\displaystyle\lim_{x\to\pi/6}\frac{\cos(2x) - (1/2)}{x - \pi/6}$ **16.** $\displaystyle\lim_{x\to -a}\frac{(1/x^2) - (1/a^2)}{x + a}$

In Exercises 17–24, express the derivatives of the given functions in terms of the derivatives f' and g' of the differentiable functions f and g.

17. $f(3 - x^2)$ **18.** $[f(\sqrt{x})]^2$

19. $f(2x)\sqrt{g(x/2)}$ **20.** $\dfrac{f(x) - g(x)}{f(x) + g(x)}$

21. $f(x + (g(x))^2)$ **22.** $f\left(\dfrac{g(x^2)}{x}\right)$

23. $f(\sin x)\,g(\cos x)$ **24.** $\sqrt{\dfrac{\cos f(x)}{\sin g(x)}}$

25. Find the tangent to the curve $x^3y + 2xy^3 = 12$ at the point $(2, 1)$.

26. Find the slope of the curve $3\sqrt{2}x\sin(\pi y) + 8y\cos(\pi x) = 2$ at the point $\left(\frac{1}{3}, \frac{1}{4}\right)$.

Find the indefinite integrals in Exercises 27–30.

27. $\displaystyle\int \frac{1 + x^4}{x^2}\,dx$ **28.** $\displaystyle\int \frac{1 + x}{\sqrt{x}}\,dx$

29. $\displaystyle\int \frac{2 + 3\sin x}{\cos^2 x}\,dx$ **30.** $\displaystyle\int (2x + 1)^4\,dx$

31. Find $f(x)$ given that $f'(x) = 12x^2 + 12x^3$ and $f(1) = 0$.

32. Find $g(x)$ if $g'(x) = \sin(x/3) + \cos(x/6)$ and the graph of g passes through the point $(\pi, 2)$.

33. Differentiate $x\sin x + \cos x$ and $x\cos x - \sin x$, and use the results to find the indefinite integrals

$$I_1 = \int x\cos x\,dx \quad\text{and}\quad I_2 = \int x\sin x\,dx.$$

34. Suppose that $f'(x) = f(x)$ for every x, and let $g(x) = x\,f(x)$. Calculate the first several derivatives of g and guess a formula for the nth-order derivative $g^{(n)}(x)$. Verify your guess by induction.

35. Find an equation of the straight line that passes through the origin and is tangent to the curve $y = x^3 + 2$.

36. Find an equation of the straight lines that pass through the point $(0, 1)$ and are tangent to the curve $y = \sqrt{2 + x^2}$.

37. Show that $\dfrac{d}{dx}\left(\sin^n x\sin(nx)\right) = n\sin^{n-1}x\sin((n+1)x)$. At what points x in $[0, \pi]$ does the graph of $y = \sin^n x\sin(nx)$ have a horizontal tangent. Assume that $n \ge 2$.

38. Find differentiation formulas for $y = \sin^n x\cos(nx)$, $y = \cos^n x\sin(nx)$, and $y = \cos^n x\cos(nx)$ analogous to the one given for $y = \sin^n x\sin(nx)$ in Exercise 37.

39. Let Q be the point $(0, 1)$. Find all points P on the curve $y = x^2$ such that the line PQ is normal to $y = x^2$ at P. What is the shortest distance from Q to the curve $y = x^2$?

40. **(Average and marginal profit)** Figure 2.42 shows the graph of the profit $\$P(x)$ realized by a grain exporter from its sale of x tonnes of wheat. Thus, the average profit per tonne is $\$P(x)/x$. Show that the maximum average profit occurs when the average profit equals the marginal profit. What is the geometric significance of this fact in the figure?

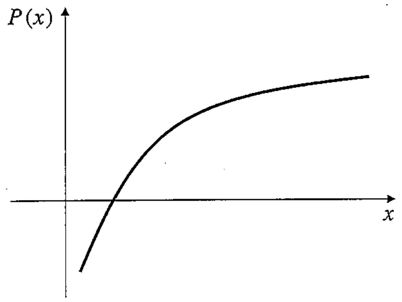

Figure 2.42

41. **(Gravitational attraction)** The gravitational attraction of the earth on a mass m at distance r from the centre of the earth is a continuous function $F(r)$ given for $r \ge 0$ by

$$F(r) = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \ge R \\ mkr & \text{if } 0 \le r < R \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

(a) Find the constant k in terms of g and R.

(b) F decreases as m moves away from the surface of the earth, either upward or downward. Show that F decreases as r increases from R at twice the rate at which F decreases as r decreases from R.

42. **(Compressibility of a gas)** The isothermal compressibility of a gas is the relative rate of change of the volume V with respect to the pressure P, at a constant temperature T, that is, $(1/V)\,dV/dP$. For a sample of an ideal gas, the temperature, pressure, and volume satisfy the equation $PV = kT$, where k is a constant related to the number of molecules of gas present in the sample. Show that the isothermal compressibility of such a gas is the negative reciprocal of the pressure:

$$\frac{1}{V}\frac{dV}{dP} = -\frac{1}{P}.$$

43. A ball is thrown upward with an initial speed of 10 m/s from the top of a building. A second ball is thrown upward with an initial speed of 20 m/s from the ground. Both balls achieve the same maximum height above the ground. How tall is the building?

44. A ball is dropped from the top of a 60 m high tower at the same instant that a second ball is thrown upward from the ground at the base of the tower. The balls collide at a height of 30 m above the ground. With what initial velocity was the second ball thrown? How fast is each ball moving when they collide?

45. **(Braking distance)** A car's brakes can decelerate the car at 20 ft/s^2. How fast can the car travel if it must be able to stop in a distance of 160 ft?

46. **(Measuring variations in g)** The period P of a pendulum of length L is given by $P = 2\pi\sqrt{L/g}$, where g is the acceleration of gravity.

(a) Assuming that L remains fixed, show that a 1% increase in g results in approximately a 1/2% decrease in the period P. (Variations in the period of a pendulum can be used to detect small variations in g from place to place on the earth's surface.)

(b) For fixed g, what percentage change in L will produce a 1% increase in P?

Challenging Problems

1. René Descartes, the inventor of analytic geometry, calculated the tangent to a parabola (or a circle or other quadratic curve) at a given point (x_0, y_0) on the curve by looking for a straight line through (x_0, y_0) having only one intersection with the given curve. Illustrate his method by writing the equation of a line through (a, a^2), having arbitrary slope m, and then finding the value of m for which the line has only one intersection with the parabola $y = x^2$. Why does the method not work for more general curves?

2. Given that $f'(x) = 1/x$ and $f(2) = 9$, find:

(a) $\displaystyle\lim_{x\to 2}\frac{f(x^2+5)-f(9)}{x-2}$ (b) $\displaystyle\lim_{x\to 2}\frac{\sqrt{f(x)}-3}{x-2}$

3. Suppose that $f'(4) = 3$, $g'(4) = 7$, $g(4) = 4$, and $g(x) \ne 4$ for $x \ne 4$. Find:

(a) $\displaystyle\lim_{x\to 4}\big(f(x)-f(4)\big)$ (b) $\displaystyle\lim_{x\to 4}\frac{f(x)-f(4)}{x^2-16}$

(c) $\displaystyle\lim_{x\to 4}\frac{f(x)-f(4)}{\sqrt{x}-2}$ (d) $\displaystyle\lim_{x\to 4}\frac{f(x)-f(4)}{(1/x)-(1/4)}$

(e) $\displaystyle\lim_{x\to 4}\frac{f(x)-f(4)}{g(x)-4}$ (f) $\displaystyle\lim_{x\to 4}\frac{f(g(x))-f(4)}{x-4}$

4. Let $f(x) = \begin{cases} x & \text{if } x = 1,\ 1/2,\ 1/3,\ 1/4,\ \dots \\ x^2 & \text{otherwise.} \end{cases}$

(a) Find all points at which f is continuous. In particular, is it continuous at $x = 0$?

(b) Is the following statement true or false? Justify your answer. For any two real numbers a and b, there is some x between a and b such that $f(x) = (f(a) + f(b))/2$.

(c) Find all points at which f is differentiable. In particular, is it differentiable at $x = 0$?

5. Suppose $f(0) = 0$ and $|f(x)| > \sqrt{|x|}$ for all x. Show that $f'(0)$ does not exist.

6. Suppose that f is a function satisfying the following conditions: $f'(0) = k$, $f(0) \ne 0$, and $f(x+y) = f(x)f(y)$ for all x and y. Show that $f(0) = 1$ and that $f'(x) = k\,f(x)$ for every x. (We will study functions with these properties in Chapter 3.)

7. Suppose the function g satisfies the conditions: $g'(0) = k$, and $g(x+y) = g(x) + g(y)$ for all x and y. Show that:

(a) $g(0) = 0$, (b) $g'(x) = k$ for all x, and

(c) $g(x) = kx$ for all x. *Hint:* Let $h(x) = g(x) - g'(0)x$.

8. (a) If f is differentiable at x, show that

(i) $\displaystyle\lim_{h\to 0}\frac{f(x)-f(x-h)}{h} = f'(x)$

(ii) $\lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h} = f'(x)$

(b) Show that the existence of the limit in (i) guarantees that f is differentiable at x.

(c) Show that the existence of the limit in (ii) does *not* guarantee that f is differentiable at x. *Hint:* Consider the function $f(x) = |x|$ at $x = 0$.

9. Show that there is a line through $(a, 0)$ that is tangent to the curve $y = x^3$ at $x = 3a/2$. If $a \neq 0$, is there any other line through $(a, 0)$ that is tangent to the curve? If (x_0, y_0) is an arbitrary point, what is the maximum number of lines through (x_0, y_0) that can be tangent to $y = x^3$? the minimum number?

10. Make a sketch showing that there are two straight lines, each of which is tangent to both of the parabolas $y = x^2 + 4x + 1$ and $y = -x^2 + 4x - 1$. Find equations of the two lines.

11. Show that if $b > 1/2$, there are three straight lines through $(0, b)$, each of which is normal to the curve $y = x^2$. How many such lines are there if $b = 1/2$? if $b < 1/2$?

12. (Distance from a point to a curve) Find the point on the curve $y = x^2$ that is closest to the point $(3, 0)$. *Hint:* The line from $(3, 0)$ to the closest point Q on the parabola is normal to the parabola at Q.

13. (Envelope of a family of lines) Show that for each value of the parameter m, the line $y = mx - (m^2/4)$ is tangent to the parabola $y = x^2$. (The parabola is called the *envelope* of the family of lines $y = mx - (m^2/4)$.) Find $f(m)$ such that the family of lines $y = mx + f(m)$ has envelope the parabola $y = Ax^2 + Bx + C$.

14. (Common tangents) Consider the two parabolas with equations $y = x^2$ and $y = Ax^2 + Bx + C$. We assume that $A \neq 0$, and if $A = 1$, then either $B \neq 0$ or $C \neq 0$, so that the two equations do represent different parabolas. Show that:

(a) the two parabolas are tangent to each other if $B^2 = 4C(A - 1)$;

(b) the parabolas have two common tangent lines if and only if $A \neq 1$ and $A\big(B^2 - 4C(A-1)\big) > 0$;

(c) the parabolas have exactly one common tangent line if either $A = 1$ and $B \neq 0$, or $A \neq 1$ and $B^2 = 4C(A - 1)$;

(d) the parabolas have no common tangent lines if either $A = 1$ and $B = 0$, or $A \neq 1$ and $A\big(B^2 - 4C(A-1)\big) < 0$.

Make sketches illustrating each of the above possibilities.

15. Let C be the graph of $y = x^3$.

(a) Show that if $a \neq 0$, then the tangent to C at $x = a$ also intersects C at a second point $x = b$.

(b) Show that the slope of C at $x = b$ is four times its slope at $x = a$.

(c) Can any line be tangent to C at more than one point?

(d) Can any line be tangent to the graph of $y = Ax^3 + Bx^2 + Cx + D$ at more than one point?

16. Let C be the graph of $y = x^4 - 2x^2$.

(a) Find all horizontal lines that are tangent to C.

(b) One of the lines found in (a) is tangent to C at two different points. Show that there are no other lines with this property.

(c) Find an equation of a straight line that is tangent to the graph of $y = x^4 - 2x^2 + x$ at two different points. Can there exist more than one such line? Why?

17. (Double tangents) A line tangent to the quartic (fourth-degree polynomial) curve C with equation $y = ax^4 + bx^3 + cx^2 + dx + e$ at $x = p$ may intersect C at zero, one, or two other points. If it meets C at only one other point $x = q$, it must be tangent to C at that point also, and it is thus a "double tangent."

(a) Find the condition that must be satisfied by the coefficients of the quartic to ensure that there does exist such a double tangent, and show that there cannot be more than one such double tangent. Illustrate this by applying your results to $y = x^4 - 2x^2 + x - 1$.

(b) If the line PQ is tangent to C at two distinct points $x = p$ and $x = q$, show that PQ is parallel to the line tangent to C at $x = (p + q)/2$.

(c) If the line PQ is tangent to C at two distinct points $x = p$ and $x = q$, show that C has two distinct inflection points R and S and that RS is parallel to PQ.

18. Verify the following formulas for every positive integer n:

(a) $\dfrac{d^n}{dx^n} \cos(ax) = a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$

(b) $\dfrac{d^n}{dx^n} \sin(ax) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$

(c) $\dfrac{d^n}{dx^n}\left(\cos^4 x + \sin^4 x\right) = 4^{n-1} \cos\left(4x + \dfrac{n\pi}{2}\right)$

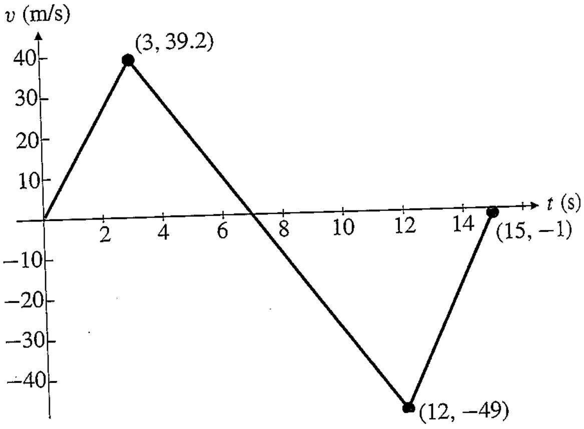

Figure 2.43

19. (Rocket with a parachute) A rocket is fired from the top of a tower at time $t = 0$. It experiences constant upward acceleration until its fuel is used up. Thereafter its acceleration is the constant downward acceleration of gravity until, during its fall, it deploys a parachute that gives it a constant upward acceleration again to slow it down. The rocket hits the ground near the base of the tower. The upward velocity v (in metres per second) is graphed against time in Figure 2.43. From information in the figure answer the following questions:

(a) How long did the fuel last?

(b) When was the rocket's height maximum?

(c) When was the parachute deployed?

(d) What was the rocket's upward acceleration while its motor was firing?

(e) What was the maximum height achieved by the rocket?

(f) How high was the tower from which the rocket was fired?