EXAMPLE 11 Show that if $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Squeeze Theorem.

EXERCISES 1.2

1. Find: (a) $\lim_{x\to -1} f(x)$, (b) $\lim_{x\to 0} f(x)$, and (c) $\lim_{x\to 1} f(x)$, for the function f whose graph is shown in Figure 1.12.

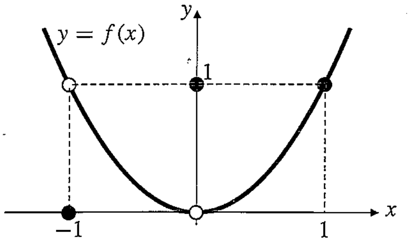

Figure 1.12

2. For the function $y = g(x)$ graphed in Figure 1.13, find each of the following limits or explain why it does not exist.
(a) $\lim_{x\to 1} g(x)$, (b) $\lim_{x\to 2} g(x)$, (c) $\lim_{x\to 3} g(x)$

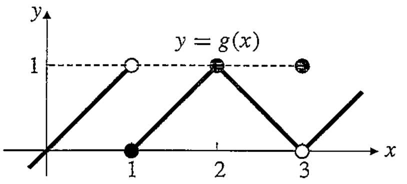

Figure 1.13

In Exercises 3–6, find the indicated one-sided limit of the function g whose graph is given in Figure 1.13.

3. $\lim_{x\to 1-} g(x)$ **4.** $\lim_{x\to 1+} g(x)$

5. $\lim_{x\to 3+} g(x)$ **6.** $\lim_{x\to 3-} g(x)$

In Exercises 7–36, evaluate the limit or explain why it does not exist.

7. $\lim_{x\to 4} (x^2 - 4x + 1)$ **8.** $\lim_{x\to 2} 3(1 - x)(2 - x)$

9. $\lim_{x\to 3} \dfrac{x+3}{x+6}$ **10.** $\lim_{t\to -4} \dfrac{t^2}{4-t}$

11. $\lim_{x\to 1} \dfrac{x^2-1}{x+1}$ **12.** $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

13. $\lim_{x\to 3} \dfrac{x^2-6x+9}{x^2-9}$ **14.** $\lim_{x\to -2} \dfrac{x^2+2x}{x^2-4}$

15. $\lim_{h\to 2} \dfrac{1}{4-h^2}$ **16.** $\lim_{h\to 0} \dfrac{3h+4h^2}{h^2-h^3}$

17. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ **18.** $\lim_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$

19. $\lim_{x\to \pi} \dfrac{(x-\pi)^2}{\pi x}$ **20.** $\lim_{x\to -2} |x-2|$

21. $\lim_{x\to 0} \dfrac{|x-2|}{x-2}$ **22.** $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$

23. $\lim_{t\to 1} \dfrac{t^2-1}{t^2-2t+1}$ **24.** $\lim_{x\to 2} \dfrac{\sqrt{4-4x+x^2}}{x-2}$

25. $\lim_{t\to 0} \dfrac{t}{\sqrt{4+t}-\sqrt{4-t}}$ **26.** $\lim_{x\to 1} \dfrac{x^2-1}{\sqrt{x+3}-2}$

27. $\lim_{t\to 0} \dfrac{t^2+3t}{(t+2)^2-(t-2)^2}$ **28.** $\lim_{s\to 0} \dfrac{(s+1)^2-(s-1)^2}{s}$

29. $\lim_{y\to 1} \dfrac{y-4\sqrt{y}+3}{y^2-1}$ **30.** $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$

31. $\lim_{x\to 2} \dfrac{x^4-16}{x^3-8}$ **32.** $\lim_{x\to 8} \dfrac{x^{2/3}-4}{x^{1/3}-2}$

33. $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right)$ **34.** $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{1}{x^2-4}\right)$

35. $\lim_{x\to 0} \dfrac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}$ **36.** $\lim_{x\to 0} \dfrac{|3x-1|-|3x+1|}{x}$

The limit $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ occurs frequently in the study of calculus. (Can you guess why?) Evaluate this limit for the functions f in Exercises 37–42.

37. $f(x) = x^2$ **38.** $f(x) = x^3$

39. $f(x) = \dfrac{1}{x}$ **40.** $f(x) = \dfrac{1}{x^2}$

41. $f(x) = \sqrt{x}$ **42.** $f(x) = 1/\sqrt{x}$

Examine the graphs of $\sin x$ and $\cos x$ in Section P.7 to determine the limits in Exercises 43–46.

43. $\lim_{x\to \pi/2} \sin x$ **44.** $\lim_{x\to \pi/4} \cos x$

45. $\lim_{x\to \pi/3} \cos x$ **46.** $\lim_{x\to 2\pi/3} \sin x$

47. Make a table of values of $f(x) = (\sin x)/x$ for a sequence of values of x approaching 0, say ± 1.0, ± 0.1, ± 0.01, ± 0.001, ± 0.0001, and ± 0.00001. Make sure your calculator is set in *radian mode* rather than degree mode. Guess the value of $\lim_{x\to 0} f(x)$.

48. Repeat Exercise 47 for $f(x) = \dfrac{1-\cos x}{x^2}$.

In Exercises 49–60, find the indicated one-sided limit or explain why it does not exist.

49. $\lim_{x\to 2-} \sqrt{2-x}$ **50.** $\lim_{x\to 2+} \sqrt{2-x}$

51. $\lim_{x\to -2-} \sqrt{2-x}$

52. $\lim_{x\to -2+} \sqrt{2-x}$

53. $\lim_{x\to 0} \sqrt{x^3-x}$

54. $\lim_{x\to 0-} \sqrt{x^3-x}$

55. $\lim_{x\to 0+} \sqrt{x^3-x}$

56. $\lim_{x\to 0+} \sqrt{x^2-x^4}$

57. $\lim_{x\to a-} \dfrac{|x-a|}{x^2-a^2}$

58. $\lim_{x\to a+} \dfrac{|x-a|}{x^2-a^2}$

59. $\lim_{x\to 2-} \dfrac{x^2-4}{|x+2|}$

60. $\lim_{x\to 2+} \dfrac{x^2-4}{|x+2|}$

Exercises 61–64 refer to the function

$$f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0. \end{cases}$$

Find the indicated limits.

61. $\lim_{x\to -1-} f(x)$

62. $\lim_{x\to -1+} f(x)$

63. $\lim_{x\to 0+} f(x)$

64. $\lim_{x\to 0-} f(x)$

65. Suppose $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = -3$. Find:

(a) $\lim_{x\to 4}\big(g(x)+3\big)$ (b) $\lim_{x\to 4} xf(x)$

(c) $\lim_{x\to 4}\big(g(x)\big)^2$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$.

66. Suppose $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = -2$. Find:

(a) $\lim_{x\to a}\big(f(x)+g(x)\big)$ (b) $\lim_{x\to a} f(x)\cdot g(x)$

(c) $\lim_{x\to a} 4g(x)$ (d) $\lim_{x\to a} f(x)/g(x)$.

67. If $\lim_{x\to 2} \dfrac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$.

68. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = -2$, find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} \dfrac{f(x)}{x}$.

Using Graphing Utilities to Find Limits

Graphing calculators or computer software can be used to evaluate limits at least approximately. Simply "zoom" the plot window to show smaller and smaller parts of the graph near the point where the limit is to be found. Find the following limits by graphical techniques. Where you think it justified, give an exact answer. Otherwise, give the answer correct to 4 decimal places. Remember to ensure that your calculator or software is set for radian mode when using trigonometric functions.

69. $\lim_{x\to 0} \dfrac{\sin x}{x}$

70. $\lim_{x\to 0} \dfrac{\sin(2\pi x)}{\sin(3\pi x)}$

71. $\lim_{x\to 1-} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

72. $\lim_{x\to 0+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

73. On the same graph plot the three functions $y = x\sin(1/x)$, $y = x$, and $y = -x$ for $-0.2 \le x \le 0.2$, $-0.2 \le y \le 0.2$. Describe the behaviour of $f(x) = x\sin(1/x)$ near $x = 0$. Does $\lim_{x\to 0} f(x)$ exist, and if so, what is its value? Could you have predicted this before drawing the graph? Why?

Using the Squeeze Theorem

74. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

75. If $2-x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

76. (a) Sketch the curves $y = x^2$ and $y = x^4$ on the same graph. Where do they intersect?

(b) The function $f(x)$ satisfies:

$$\begin{cases} x^2 \le f(x) \le x^4 & \text{if } x < -1 \text{ or } x > 1 \\ x^4 \le f(x) \le x^2 & \text{if } -1 \le x \le 1 \end{cases}$$

Find (i) $\lim_{x\to -1} f(x)$, (ii) $\lim_{x\to 0} f(x)$, (iii) $\lim_{x\to 1} f(x)$.

77. On what intervals is $x^{1/3} < x^3$? On what intervals is $x^{1/3} > x^3$? If the graph of $y = h(x)$ always lies between the graphs of $y = x^{1/3}$ and $y = x^3$, for what real numbers a can you determine the value of $\lim_{x\to a} h(x)$? Find the limit for each of these values of a.

78. What is the domain of $x \sin\dfrac{1}{x}$? Evaluate $\lim_{x\to 0} x\sin\dfrac{1}{x}$.

79. Suppose $|f(x)| \le g(x)$ for all x. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 3$?

1.3 Limits at Infinity and Infinite Limits

In this section we will extend the concept of limit to allow for two situations not covered by the definitions of limit and one-sided limit in the previous section:

(i) limits at infinity, where x becomes arbitrarily large, positive or negative;

(ii) infinite limits, which are not really limits at all but provide useful symbolism for describing the behaviour of functions whose values become arbitrarily large, positive or negative.

```
> limit((x^2-a^2)/(abs(x-a)),x=a,right);
```

$$2a$$

Finally we use Maple to confirm the limit discussed in Example 2 in Section 1.2

```
> limit((1+x^2)^(1/x^2),x=0); evalf(%);
```

$$e$$

$$2.718281828$$

We will learn a great deal about this very important number in Chapter 3.

EXERCISES 1.3

Find the limits in Exercises 1–10.

1. $\lim\limits_{x\to\infty} \dfrac{x}{2x-3}$

2. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-4}$

3. $\lim\limits_{x\to\infty} \dfrac{3x^3-5x^2+7}{8+2x-5x^3}$

4. $\lim\limits_{x\to-\infty} \dfrac{x^2-2}{x-x^2}$

5. $\lim\limits_{x\to-\infty} \dfrac{x^2+3}{x^3+2}$

6. $\lim\limits_{x\to\infty} \dfrac{x^2+\sin x}{x^2+\cos x}$

7. $\lim\limits_{x\to\infty} \dfrac{3x+2\sqrt{x}}{1-x}$

8. $\lim\limits_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$

9. $\lim\limits_{x\to-\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$

10. $\lim\limits_{x\to-\infty} \dfrac{2x-5}{|3x+2|}$

In Exercises 11–34 evaluate the indicated limit. If it does not exist, is the limit ∞, $-\infty$, or neither?

11. $\lim\limits_{x\to3} \dfrac{1}{3-x}$

12. $\lim\limits_{x\to3} \dfrac{1}{(3-x)^2}$

13. $\lim\limits_{x\to3-} \dfrac{1}{3-x}$

14. $\lim\limits_{x\to3+} \dfrac{1}{3-x}$

15. $\lim\limits_{x\to-5/2} \dfrac{2x+5}{5x+2}$

16. $\lim\limits_{x\to-2/5} \dfrac{2x+5}{5x+2}$

17. $\lim\limits_{x\to-(2/5)-} \dfrac{2x+5}{5x+2}$

18. $\lim\limits_{x\to-(2/5)+} \dfrac{2x+5}{5x+2}$

19. $\lim\limits_{x\to2+} \dfrac{x}{(2-x)^3}$

20. $\lim\limits_{x\to1-} \dfrac{x}{\sqrt{1-x^2}}$

21. $\lim\limits_{x\to1+} \dfrac{1}{|x-1|}$

22. $\lim\limits_{x\to1-} \dfrac{1}{|x-1|}$

23. $\lim\limits_{x\to2} \dfrac{x-3}{x^2-4x+4}$

24. $\lim\limits_{x\to1+} \dfrac{\sqrt{x^2-x}}{x-x^2}$

25. $\lim\limits_{x\to\infty} \dfrac{x+x^3+x^5}{1+x^2+x^3}$

26. $\lim\limits_{x\to\infty} \dfrac{x^3+3}{x^2+2}$

27. $\lim\limits_{x\to\infty} \dfrac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

28. $\lim\limits_{x\to\infty} \left(\dfrac{x^2}{x+1}-\dfrac{x^2}{x-1}\right)$

29. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

30. $\lim\limits_{x\to\infty} (\sqrt{x^2+2x}-\sqrt{x^2-2x})$

31. $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

32. $\lim\limits_{x\to-\infty} \dfrac{1}{\sqrt{x^2+2x}-x}$

33. What are the horizontal asymptotes of $y = \dfrac{1}{\sqrt{x^2-2x}-x}$? What are its vertical asymptotes?

34. What are the horizontal and vertical asymptotes of $y = \dfrac{2x-5}{|3x+2|}$?

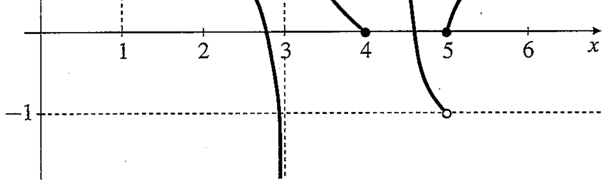

Figure 1.18

The function f whose graph is shown in Figure 1.18 has domain $[0, \infty)$. Find the limits of f indicated in Exercises 35–45.

35. $\lim\limits_{x\to0+} f(x)$

36. $\lim\limits_{x\to1} f(x)$

37. $\lim\limits_{x\to2+} f(x)$

38. $\lim\limits_{x\to2-} f(x)$

39. $\lim\limits_{x\to3-} f(x)$

40. $\lim\limits_{x\to3+} f(x)$

41. $\lim\limits_{x\to4+} f(x)$

42. $\lim\limits_{x\to4-} f(x)$

43. $\lim\limits_{x\to5-} f(x)$

44. $\lim\limits_{x\to5+} f(x)$

45. $\lim\limits_{x\to\infty} f(x)$

46. What asymptotes does the graph in Figure 1.18 have?

Exercises 47–52 refer to the **greatest integer function** $\lfloor x \rfloor$ graphed in Figure 1.19. Find the indicated limit or explain why it does not exist.

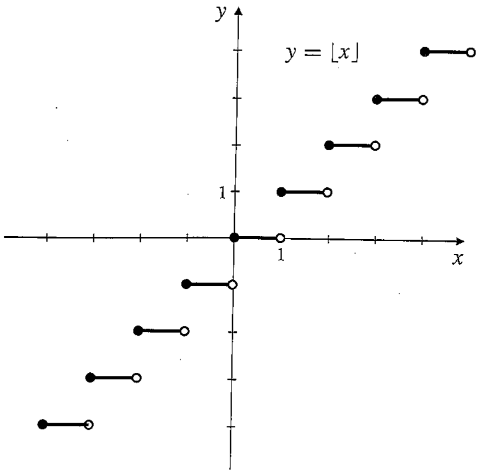

Figure 1.19

47. $\lim_{x \to 3+} \lfloor x \rfloor$

48. $\lim_{x \to 3-} \lfloor x \rfloor$

49. $\lim_{x \to 3} \lfloor x \rfloor$

50. $\lim_{x \to 2.5} \lfloor x \rfloor$

51. $\lim_{x \to 0+} \lfloor 2 - x \rfloor$

52. $\lim_{x \to -3-} \lfloor x \rfloor$

53. Parking in a certain parking lot costs \$1.50 for each hour or part of an hour. Sketch the graph of the function $C(t)$ representing the cost of parking for t hours. At what values of t does $C(t)$ have a limit? Evaluate $\lim_{t \to t_0-} C(t)$ and $\lim_{t \to t_0+} C(t)$ for an arbitrary number $t_0 > 0$.

54. If $\lim_{x \to 0+} f(x) = L$, find $\lim_{x \to 0-} f(x)$ if (a) f is even, (b) f is odd.

55. If $\lim_{x \to 0+} f(x) = A$ and $\lim_{x \to 0-} f(x) = B$, find

(a) $\lim_{x \to 0+} f(x^3 - x)$

(b) $\lim_{x \to 0-} f(x^3 - x)$

(c) $\lim_{x \to 0-} f(x^2 - x^4)$

(d) $\lim_{x \to 0+} f(x^2 - x^4)$.

1.4 Continuity

When a car is driven along a highway, its distance from its starting point depends on time in a *continuous* way, changing by small amounts over short intervals of time. But not all quantities change in this way. When the car is parked in a parking lot where the rate is quoted as "\$2.00 per hour or portion," the parking charges remain at \$2.00 for the first hour and then suddenly jump to \$4.00 as soon as the first hour has passed. The function relating parking charges to parking time will be called *discontinuous* at each hour. In this section we will define continuity and show how to tell whether a function is continuous. We will also examine some important properties possessed by continuous functions.

Continuity at a Point

Most functions that we encounter have domains that are intervals, or unions of separate intervals. A point P in the domain of such a function is called an **interior point** of the domain if it belongs to some *open* interval contained in the domain. If it is not an interior point, then P is called an **endpoint** of the domain. For example, the domain of the function $f(x) = \sqrt{4 - x^2}$ is the closed interval $[-2, 2]$, which consists of interior points in the interval $(-2, 2)$, a left endpoint -2, and a right endpoint 2. The domain of the function $g(x) = 1/x$ is the union of open intervals $(-\infty, 0) \cup (0, \infty)$ and consists entirely of interior points. Note that although 0 is an endpoint of each of those intervals, it does not belong to the domain of g and so is not an endpoint of that domain.

DEFINITION 4

Continuity at an interior point

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

If either $\lim_{x \to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

In graphical terms, f is continuous at an interior point c of its domain if its graph has no break in it at the point $(c, f(c))$; in other words, if you can draw the graph through

Therefore, $N^2 \le N$ and so $N^2 - N \le 0$.
Thus, $N(N-1) \le 0$ and we must have $N - 1 \le 0$.
Therefore, $N \le 1$. Since also $N \ge 1$, we have $N = 1$.
Therefore, 1 is the largest positive integer.

The only error we have made here is in the assumption (in the first line) that the problem has a solution. It is partly to avoid logical pitfalls like this that mathematicians prove existence theorems.

EXERCISES 1.4

Exercises 1–3 refer to the function g defined on $[-2, 2]$, whose graph is shown in Figure 1.34.

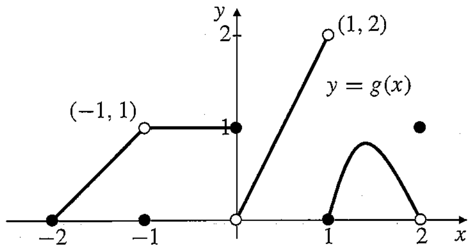

Figure 1.34

1. State whether g is (a) continuous, (b) left continuous, (c) right continuous, and (d) discontinuous at each of the points -2, -1, 0, 1, and 2.
2. At what points in its domain does g have a removable discontinuity, and how should g be redefined at each of those points so as to be continuous there?
3. Does g have an absolute maximum value on $[-2, 2]$? an absolute minimum value?

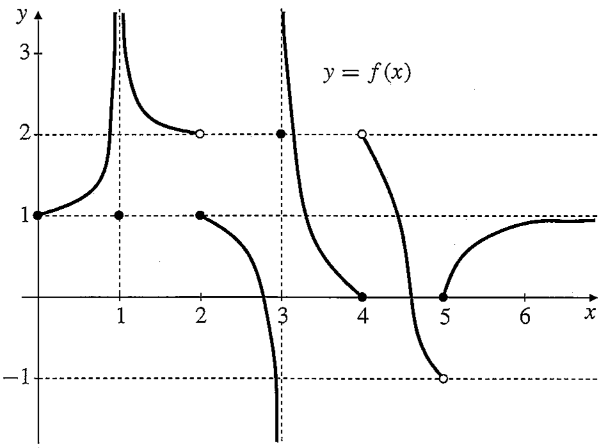

Figure 1.35

4. At what points is the function f, whose graph is shown in Figure 1.35, discontinuous? At which of those points is it left continuous? right continuous?
5. Can the function f graphed in Figure 1.35 be redefined at the single point $x = 1$ so that it becomes continuous there?
6. The function $\operatorname{sgn}(x) = x/|x|$ is neither continuous nor discontinuous at $x = 0$. How is this possible?

In Exercises 7–12, state where in its domain the given function is continuous, where it is left or right continuous, and where it is just discontinuous.

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

10. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 0.987 & \text{if } x > 1 \end{cases}$

11. The least integer function $\lceil x \rceil$ of Example 11 in Section P.5.
12. The cost function $C(t)$ of Exercise 53 in Section 1.3.

In Exercises 13–16, how should the given function be defined at the given point to be continuous there? Give a formula for the continuous extension to that point.

13. $\dfrac{x^2-4}{x-2}$ at $x = 2$

14. $\dfrac{1+t^3}{1-t^2}$ at $t = -1$

15. $\dfrac{t^2-5t+6}{t^2-t-6}$ at 3

16. $\dfrac{x^2-2}{x^4-4}$ at $\sqrt{2}$

17. Find k so that $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ k - x^2 & \text{if } x > 2 \end{cases}$ is a continuous function.

18. Find m so that $g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \ge 3 \end{cases}$ is continuous for all x.

19. Does the function x^2 have a maximum value on the open interval $-1 < x < 1$? a minimum value? Explain.
20. The Heaviside function of Example 1 has both absolute maximum and minimum values on the interval $[-1, 1]$, but it is not continuous on that interval. Does this violate the Max-Min Theorem? Why?

Exercises 21–24 ask for maximum and minimum values of functions. They can all be done by the method of Example 9.

21. The sum of two nonnegative numbers is 8. What is the largest possible value of their product?
22. The sum of two nonnegative numbers is 8. What is (a) the smallest and (b) the largest possible value for the sum of their squares?
23. A software company estimates that if it assigns x programmers to work on the project, it can develop a new product in T days, where

$$T = 100 - 30x + 3x^2.$$

How many programmers should the company assign in order to complete the development as quickly as possible?

24. It costs a desk manufacturer $\$(245x - 30x^2 + x^3)$ to send a shipment of x desks to its warehouse. How many desks should it include in each shipment to minimize the average shipping cost per desk?

Find the intervals on which the functions $f(x)$ in Exercises 25–28 are positive and negative.

25. $f(x) = \dfrac{x^2-1}{x}$

26. $f(x) = x^2 + 4x + 3$

27. $f(x) = \dfrac{x^2-1}{x^2-4}$

28. $f(x) = \dfrac{x^2+x-2}{x^3}$

29. Show that $f(x) = x^3 + x - 1$ has a zero between $x = 0$ and $x = 1$.

30. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

31. Show that the function $F(x) = (x-a)^2(x-b)^2 + x$ has the value $(a+b)/2$ at some point x.

32. **(A fixed-point theorem)** Suppose that f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$. (c is called a fixed point of the function f.) *Hint:* If $f(0) = 0$ or $f(1) = 1$, you are done. If not, apply the Intermediate-Value Theorem to $g(x) = f(x) - x$.

33. If an even function f is right continuous at $x = 0$, show that it is continuous at $x = 0$.

34. If an odd function f is right continuous at $x = 0$, show that it is continuous at $x = 0$ and that it satisfies $f(0) = 0$.

Use a graphing utility to find maximum and minimum values of the functions in Exercises 35–38 and the points x where they occur. Obtain 3 decimal place accuracy for all answers.

35. $f(x) = \dfrac{x^2-2x}{x^4+1}$ on $[-5, 5]$

36. $f(x) = \dfrac{\sin x}{6+x}$ on $[-\pi, \pi]$

37. $f(x) = x^2 + \dfrac{4}{x}$ on $[1, 3]$

38. $f(x) = \sin(\pi x) + x(\cos(\pi x) + 1)$ on $[0, 1]$

Use a graphing utility or a programmable calculator and the Bisection Method to solve the equations in Exercises 39–40 to 3 decimal places. As a first step, try to guess a small interval that you can be sure contains a root.

39. $x^3 + x - 1 = 0$

40. $\cos x - x = 0$

Use Maple's `fsolve` routine to solve the equations in 41–42.

41. $\sin x + 1 - x^2 = 0$ (two roots)

42. $x^4 - x - 1 = 0$ (two roots)

43. Investigate the difference between the Maple routines `fsolve(f,x)`, `solve(f,x)`, and `evalf(solve(f,x))`, where `f := x^3-x-1=0`. Note that no interval is specified for x here.

1.5 The Formal Definition of Limit

The material in this section is optional.

The *informal* definition of limit given in Section 1.2 is not precise enough to enable us to prove results about limits such as those given in Theorems 2–4 of Section 1.2. A more precise *formal* definition is based on the idea of controlling the input x of a function f so that the output $f(x)$ will lie in a specific interval.

EXAMPLE 1 The area of a circular disk of radius r cm is $A = \pi r^2$ cm^2. A machinist is required to manufacture a circular metal disk having area 400π cm^2 within an error tolerance of ± 5 cm^2. How close to 20 cm must the machinist control the radius of the disk to achieve this?

Solution The machinist wants $|\pi r^2 - 400\pi| < 5$, that is,

$$400\pi - 5 < \pi r^2 < 400\pi + 5,$$

or, equivalently,

$$\sqrt{400 - (5/\pi)} < r < \sqrt{400 + (5/\pi)}$$
$$19.96017 < r < 20.03975.$$

Thus, the machinist needs $|r - 20| < 0.03975$; she must ensure that the radius of the disk differs from 20 cm by less than 0.4 mm so that the area of the disk will lie within the required error tolerance.

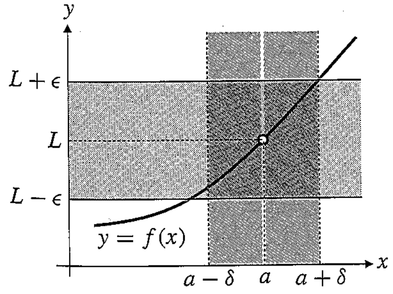

Figure 1.36 If $x \ne a$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$

When we say that $f(x)$ has limit L as x approaches a, we are really saying that we can ensure that the *error* $|f(x) - L|$ will be less than *any* allowed tolerance, no matter how small, by taking x *close enough* to a (but not equal to a). It is traditional to use ϵ, the Greek letter "epsilon," for the size of the allowable *error* and δ, the Greek letter "delta," for the *difference* $x - a$ that measures how close x must be to a to ensure that the error is within that tolerance. These are the letters that Cauchy and Weierstrass used in their pioneering work on limits and continuity in the nineteenth century.

To show that $f(x)$ has an infinite limit at a, we must ensure that $f(x)$ is larger than any given positive number (say B) by restricting x to a sufficiently small interval centred at a, and requiring that $x \neq a$.

DEFINITION 11

Infinite limits

We say that $f(x)$ approaches infinity as x approaches a and write

$$\lim_{x\to a} f(x) = \infty,$$

if for every positive number B we can find a positive number δ, possibly depending on B, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and $f(x) > B$.

Try to formulate the corresponding definition for the concept $\lim_{x\to a} f(x) = -\infty$. Then try to modify both definitions to cover the case of infinite one-sided limits and infinite limits at infinity.

EXAMPLE 7 Verify that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution Let B be any positive number. We have

$$\frac{1}{x^2} > B \qquad \text{provided that} \quad x^2 < \frac{1}{B}.$$

If $\delta = 1/\sqrt{B}$, then

$$0 < |x| < \delta \quad \Rightarrow \quad x^2 < \delta^2 = \frac{1}{B} \quad \Rightarrow \quad \frac{1}{x^2} > B.$$

Therefore $\lim_{x\to 0} 1/x^2 = \infty$.

EXERCISES 1.5

1. The length L of a metal rod is given in terms of the temperature T (°C) by $L = 39.6 + 0.025T$ cm. Within what range of temperature must the rod be kept if its length must be maintained within ± 1 mm of 40 cm?

2. What is the largest tolerable error in the 20 cm edge length of a cubical cardboard box if the volume of the box must be within $\pm 1.2\%$ of 8,000 cm^3?

In Exercises 3–6, in what interval must x be confined if $f(x)$ must be within the given distance ϵ of the number L?

3. $f(x) = 2x - 1$, $L = 3$, $\epsilon = 0.02$
4. $f(x) = x^2$, $L = 4$, $\epsilon = 0.1$
5. $f(x) = \sqrt{x}$, $L = 1$, $\epsilon = 0.1$
6. $f(x) = 1/x$, $L = -2$, $\epsilon = 0.01$

In Exercises 7–10, find a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L|$ will be less than the given number ϵ.

7. $f(x) = 3x + 1$, $a = 2$, $L = 7$, $\epsilon = 0.03$
8. $f(x) = \sqrt{2x+3}$, $a = 3$, $L = 3$, $\epsilon = 0.01$
9. $f(x) = x^3$, $a = 2$, $L = 8$, $\epsilon = 0.2$
10. $f(x) = 1/(x+1)$, $a = 0$, $L = 1$, $\epsilon = 0.05$

In Exercises 11–20, use the formal definition of limit to verify the indicated limit.

11. $\lim_{x\to 1}(3x + 1) = 4$
12. $\lim_{x\to 2}(5 - 2x) = 1$
13. $\lim_{x\to 0} x^2 = 0$
14. $\lim_{x\to 2} \frac{x-2}{1+x^2} = 0$
15. $\lim_{x\to 1/2} \frac{1-4x^2}{1-2x} = 2$
16. $\lim_{x\to -2} \frac{x^2+2x}{x+2} = -2$
17. $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$
18. $\lim_{x\to -1} \frac{x+1}{x^2-1} = -\frac{1}{2}$
19. $\lim_{x\to 1} \sqrt{x} = 1$
20. $\lim_{x\to 2} x^3 = 8$

Give formal definitions of the limit statements in Exercises 21–26.

21. $\lim_{x\to a-} f(x) = L$
22. $\lim_{x\to -\infty} f(x) = L$
23. $\lim_{x\to a} f(x) = -\infty$
24. $\lim_{x\to \infty} f(x) = \infty$
25. $\lim_{x\to a+} f(x) = -\infty$
26. $\lim_{x\to a-} f(x) = \infty$

Use formal definitions of the various kinds of limits to prove the statements in Exercises 27–30.

27. $\lim_{x\to 1+} \frac{1}{x-1} = \infty$

28. $\lim_{x\to 1-} \frac{1}{x-1} = -\infty$

29. $\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}} = 0$

30. $\lim_{x\to\infty} \sqrt{x} = \infty$

Proving Theorems with the Definition of Limit

31. Prove that limits are unique; that is, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, prove that $L = M$. *Hint:* Suppose $L \neq M$ and let $\epsilon = |L - M|/3$.

32. If $\lim_{x\to a} g(x) = M$, show that there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |g(x)| < 1 + |M|.$$

(*Hint:* Take $\epsilon = 1$ in the definition of limit.) This says that the values of $g(x)$ are **bounded** near a point where g has a limit.

33. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, prove that $\lim_{x\to a} f(x)g(x) = LM$ (the Product Rule part of Theorem 2). *Hint:* Reread Example 4. Let $\epsilon > 0$ and write

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |g(x)||f(x) - L| + |L||g(x) - M| \end{aligned}$$

Now try to make each term in the last line less than $\epsilon/2$ by taking x close enough to a. You will need the result of Exercise 32.

34. If $\lim_{x\to a} g(x) = M$, where $M \neq 0$, show that there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |g(x)| > |M|/2.$$

35. If $\lim_{x\to a} g(x) = M$, where $M \neq 0$, show that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Hint: You will need the result of Exercise 34.

36. Use the facts proved in Exercises 33 and 35 to prove the Quotient Rule (part 5 of Theorem 2): if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, where $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

37. Use the definition of limit twice to prove Theorem 7 of Section 1.4; that is, if f is continuous at L and if $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f\left(\lim_{x\to c} g(x)\right).$$

38. Prove the Squeeze Theorem (Theorem 4 in Section 1.2). *Hint:* If $f(x) \le g(x) \le h(x)$, then

$$\begin{aligned} |g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &\le |g(x) - f(x)| + |f(x) - L| \\ &\le |h(x) - f(x)| + |f(x) - L| \\ &= |h(x) - L - (f(x) - L)| + |f(x) - L| \\ &\le |h(x) - L| + |f(x) - L| + |f(x) - L| \end{aligned}$$

Now you can make each term in the last expression less than $\epsilon/3$ and so complete the proof.

CHAPTER REVIEW

Key Ideas

- **What do the following statements and phrases mean?**
  - ◇ the average rate of change of $f(x)$ on $[a, b]$
  - ◇ the instantaneous rate of change of $f(x)$ at $x = a$
  - ◇ $\lim_{x\to a} f(x) = L$
  - ◇ $\lim_{x\to a+} f(x) = L, \quad \lim_{x\to a-} f(x) = L$
  - ◇ $\lim_{x\to\infty} f(x) = L, \quad \lim_{x\to -\infty} f(x) = L$
  - ◇ $\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a+} f(x) = -\infty$
  - ◇ f is continuous at c.
  - ◇ f is left (or right) continuous at c.
  - ◇ f has a continuous extension to c.
  - ◇ f is a continuous function.
  - ◇ f takes on maximum and minimum values on interval I.
  - ◇ f is bounded on interval I.
  - ◇ f has the intermediate-value property on interval I.
- **State as many "laws of limits" as you can.**
- **What properties must a function have if it is continuous and its domain is a closed, finite interval?**
- **How can you find zeros (roots) of a continuous function?**

Review Exercises

1. Find the average rate of change of x^3 over $[1, 3]$.
2. Find the average rate of change of $1/x$ over $[-2, -1]$.
3. Find the rate of change of x^3 at $x = 2$.
4. Find the rate of change of $1/x$ at $x = -3/2$.

Evaluate the limits in Exercises 5–30 or explain why they do not exist.

5. $\lim_{x\to 1} (x^2 - 4x + 7)$

6. $\lim_{x\to 2} \frac{x^2}{1 - x^2}$

7. $\lim_{x\to 1} \frac{x^2}{1-x^2}$

8. $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$

9. $\lim_{x\to 2} \frac{x^2-4}{x^2-4x+4}$

10. $\lim_{x\to 2-} \frac{x^2-4}{x^2-4x+4}$

11. $\lim_{x\to -2+} \frac{x^2-4}{x^2+4x+4}$

12. $\lim_{x\to 4} \frac{2-\sqrt{x}}{x-4}$

13. $\lim_{x\to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}}$

14. $\lim_{h\to 0} \frac{h}{\sqrt{x+3h}-\sqrt{x}}$

15. $\lim_{x\to 0+} \sqrt{x-x^2}$

16. $\lim_{x\to 0} \sqrt{x-x^2}$

17. $\lim_{x\to 1} \sqrt{x-x^2}$

18. $\lim_{x\to 1-} \sqrt{x-x^2}$

19. $\lim_{x\to \infty} \frac{1-x^2}{3x^2-x-1}$

20. $\lim_{x\to -\infty} \frac{2x+100}{x^2+3}$

21. $\lim_{x\to -\infty} \frac{x^3-1}{x^2+4}$

22. $\lim_{x\to \infty} \frac{x^4}{x^2-4}$

23. $\lim_{x\to 0+} \frac{1}{\sqrt{x-x^2}}$

24. $\lim_{x\to 1/2} \frac{1}{\sqrt{x-x^2}}$

25. $\lim_{x\to \infty} \sin x$

26. $\lim_{x\to \infty} \frac{\cos x}{x}$

27. $\lim_{x\to 0} x \sin \frac{1}{x}$

28. $\lim_{x\to 0} \sin \frac{1}{x^2}$

29. $\lim_{x\to -\infty} [x+\sqrt{x^2-4x+1}]$

30. $\lim_{x\to \infty} [x+\sqrt{x^2-4x+1}]$

At what, if any, points in its domain is the function f in Exercises 31–38 discontinuous? Is f left or right continuous at these points? In Exercises 35 and 36, H refers to the Heaviside function: $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$.

31. $f(x) = x^3 - 4x^2 + 1$

32. $f(x) = \frac{x}{x+1}$

33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \leq 2 \end{cases}$

34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \leq 1 \end{cases}$

35. $f(x) = H(x-1)$

36. $f(x) = H(9-x^2)$

37. $f(x) = |x| + |x+1|$

38. $f(x) = \begin{cases} |x|/|x+1| & \text{if } x \neq -1 \\ 1 & \text{if } x = -1 \end{cases}$

Challenging Problems

1. Show that the average rate of change of the function x^3 over the interval $[a, b]$, where $0 < a < b$, is equal to the instantaneous rate of change of x^3 at $x = \sqrt{(a^2+ab+b^2)/3}$. Is this point to the left or to the right of the midpoint $(a+b)/2$ of the interval $[a, b]$?

2. Evaluate $\lim_{x\to 0} \frac{x}{|x-1|-|x+1|}$.

3. Evaluate $\lim_{x\to 3} \frac{|5-2x|-|x-2|}{|x-5|-|3x-7|}$.

4. Evaluate $\lim_{x\to 64} \frac{x^{1/3}-4}{x^{1/2}-8}$.

5. Evaluate $\lim_{x\to 1} \frac{\sqrt{3+x}-2}{\sqrt[3]{7+x}-2}$.

6. The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$:

$$r_+(a) = \frac{-1+\sqrt{1+a}}{a} \text{ and } r_-(a) = \frac{-1-\sqrt{1+a}}{a}.$$

(a) What happens to the root $r_-(a)$ when $a \to 0$?

(b) Investigate numerically what happens to the root $r_+(a)$ when $a \to 0$ by trying the values $a = 1, \pm 0.1, \pm 0.01, \ldots$. For values such as $a = 10^{-8}$, the limited precision of your calculator may produce some interesting results. What happens, and why?

(c) Evaluate $\lim_{a\to 0} r_+(a)$ mathematically by using the identity

$$\sqrt{A} - \sqrt{B} = \frac{A-B}{\sqrt{A}+\sqrt{B}}.$$

7. TRUE or FALSE? If TRUE, give reasons; if FALSE, give a counterexample.

(a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.

(b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.

(c) If f is continuous at a, then so is $|f|$.

(d) If $|f|$ is continuous at a, then so is f.

(e) If $f(x) < g(x)$ for all x in an interval around a, and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

8. (a) If f is a continuous function defined on a closed interval $[a, b]$, show that $R(f)$ is a closed interval.

(b) What are the possibilities for $R(f)$ if $D(f)$ is an open interval (a, b)?

9. Consider the function $f(x) = \frac{x^2-1}{|x^2-1|}$. Find all points where f is not continuous. Does f have one-sided limits at those points, and if so, what are they?

10. Find the minimum value of $f(x) = 1/(x-x^2)$ on the interval $(0, 1)$. Explain how you know such a minimum value must exist.

11. (a) Suppose f is a continuous function on the interval $[0, 1]$, and $f(0) = f(1)$. Show that $f(a) = f\left(a+\frac{1}{2}\right)$ for some $a \in \left[0, \frac{1}{2}\right]$.

Hint: Let $g(x) = f\left(x+\frac{1}{2}\right) - f(x)$, and use the Intermediate-Value Theorem.

(b) If n is an integer larger than 2, show that $f(a) = f\left(a+\frac{1}{n}\right)$ for some $a \in \left[0, 1-\frac{1}{n}\right]$.