國立東華大學應用數學系　科目：高等微積分(一)
100學年度第一學期期末考試卷　　時間：**Jan. 07, 2012**

班級：____________________　姓名：____________________　學號____________________

Note that $B(x,r) = D(x,r)$ is the open ball with center x and radius r, $\text{cl}(A) = \bar{A}$ is the closure of A and $\text{bd}(A) = \partial A$ is the boundary of A. A' is the collection of all accumulation points of A.

1. (20%) Give the following definitions: Let (M, d) and (N, ρ) be two metric spaces, $f : M \to N$ be a function and $x \in M$.
   (a) M is sequentially compact;
   (b) M is path connected;
   (c) M is totally bounded;
   (d) f is continuous at x.
2. (20%)
   (a) Show that $(-\pi/2, \pi/2)$ is path connected. (Hence, by theorem, it is connected.)
   (b) Construct a continuous function f from $\mathbb{R}$ onto $(-\pi/2, \pi/2)$. (By the facts that $\mathbb{R}$ is connected and a continuous function maps connected subsets in connected subsets, $(-\pi/2, \pi/2)$ is connected.)
3. (15%) Let A be bounded on $\mathbb{R}^n$. Show that $cl(A)$ is compact.
4. (15%) Let M be an infinite set with discrete metric. Show that M is not compact.
5. (10%) Determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:
   (a) If $f : M \to N$ is a continuous function from a metric space M into another metric space N and U is open in M, then $f(U)$ is open in N;
   (b) If A is connected in $\mathbb{R}^2$, then A is open or closed.
6. (24%) Which of the following sets are compact? connected? or path connected?
   (a) $A = \{x \in \mathbb{R} | 0 \le x \le 1 \quad \text{and } x \text{ is irrational}\}$;
   (b) $B = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1\} \cup \{(x,0) \in \mathbb{R}^2 | 1 < x < 2\}$;
   (c) $C = \{(0,y) \in \mathbb{R}^2 | -1 \le y \le 1\} \cup \{(x, \sin(\pi/x)) \in \mathbb{R}^2 | 0 < x \le 1\}$;
   (d) $D = \{f(x) \in \mathbb{R} | x \ge 0\}$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous.
7. (16%) Let A_1 and A_2 be path connected in a metric space M and $A_1 \cap A_2 \neq \varnothing$. Show that $A_1 \cup A_2$ is path connected.
8. (20%)
   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x$. Prove that f is continuous.
   (b) Let U be open in $\mathbb{R}$. Show that $A = \{(x,y) \in \mathbb{R}^2 | x \in U\}$ is open.