第一題

(a)

∵ This limit is of the indeterminate form 1^{∞}

∴ Let $y = (1+2x)^{\frac{1}{x}}$

$$\Rightarrow \ln y = \frac{1}{x}\ln(1+2x)$$

$$\Rightarrow \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{1}{x}\ln(1+2x) \quad \ldots (\infty\cdot 0)$$

$$= \lim_{x\to 0}\frac{\ln(1+2x)}{x} \quad \ldots \left(\frac{0}{0}\right)$$

$$\overset{\text{L'Hôpital's rule}}{=} \lim_{x\to 0}\frac{\frac{2}{1+2x}}{1}$$

$$= \lim_{x\to 0}\frac{2}{1+2x}$$

$$= 2.$$

Then. $\lim_{x\to 0}\ln y = 2.$

$$\Rightarrow \lim_{x\to 0}(1+2x)^{\frac{1}{x}} = e^{2}.$$

(b)

Let $u = 2x + 1$. Then $\frac{du}{dx} = 2$, or $\frac{1}{2}du = dx$. Further, $x = \frac{u-1}{2}$.

$$\int x\sqrt{2x+1}\,dx$$
$$= \frac{1}{4}\int (u-1)u^{1/2}du$$
$$= \frac{1}{4}\int (u^{3/2} - u^{1/2})du$$
$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right) + C$$
$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

(c)

Let $u = e^{x} - e^{-x}$. Then $\frac{du}{dx} = e^{x} + e^{-x}$, or

$du = (e^{x} + e^{-x})dx.$

$$\int \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}dx = \int \frac{1}{e^{x} - e^{-x}}(e^{x} + e^{-x})dx$$
$$= \int \frac{1}{u}du$$
$$= \ln\left|e^{x} - e^{-x}\right| + C$$

(d)

Let $u = \sqrt{x} + 1$. Then

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}, \text{ or } 2\,du = \frac{1}{\sqrt{x}}dx.$$

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}dx = \int \frac{1}{\sqrt{x}+1} \cdot \frac{1}{\sqrt{x}}dx$$

$$= 2\int \frac{1}{u}du$$

$$= 2\ln\left|\sqrt{x}+1\right| + C$$

$$= 2\ln\left(\sqrt{x}+1\right) + C$$

(e)

$$\int_{1/3}^{1/2} \frac{e^{1/x}}{x^2}dx$$

Let $u = \frac{1}{x}$. Then $-du = \frac{1}{x^2}dx$, and the limits of integration become $\frac{1}{\frac{1}{3}} = 3$ and

$\frac{1}{\frac{1}{2}} = 2.$

$$= -\int_3^2 e^u\,du$$

$$= \int_2^3 e^u\,du$$

$$= (e^u)\Big|_2^3$$

$$= e^3 - e^2$$

第二題

$$f(x)=\int f'(x)\,dx=\int xe^{4-x^2}\,dx$$

Let $u=4-x^2$. Then $\dfrac{du}{dx}=-2x\,dx$, or

$$-\frac{1}{2}du=x\,dx.$$

$$\int xe^{4-x^2}dx=\int e^{4-x^2}\cdot x\,dx$$
$$=-\frac{1}{2}\int e^{u}\,du$$
$$=-\frac{1}{2}e^{4-x^2}+C$$

Since $y=1$ when $x=-2$,

$$1=-\frac{1}{2}e^{4-(-2)^2}+C,\text{ or}$$

$$C=\frac{3}{2}.$$

So, $f(x)=-\dfrac{1}{2}e^{4-x^2}+\dfrac{3}{2}.$

第三題

$$\frac{dy}{dx}=\frac{2-y^2}{xy}$$

Cross-multiplying gives

$xy\,dy=\left(2-y^2\right)dx$.

Multiplying both sides by $\dfrac{1}{x\left(2-y^2\right)}$

gives $\dfrac{y}{2-y^2}\,dy=\dfrac{1}{x}\,dx$.

Let $u_1=2-y^2$. Then, $\dfrac{du_1}{dy}=-2y$, or

$-\dfrac{1}{2}\,du_1=y\,dy$. Substituting,

$$-\frac{1}{2u_1}\,du_1=\frac{1}{x}\,dx$$

Integrating both sides,

$$\int-\frac{1}{2u_1}du_1=\int\frac{1}{x}dx$$

$$-\frac{1}{2}\int\frac{1}{u_1}du_1=\int\frac{1}{x}dx$$

$$-\frac{1}{2}\ln\left|u_1\right|+C_1=\ln\left|x\right|+C_2$$

$$-\frac{1}{2}\ln\left|2-y^2\right|=\ln\left|x\right|+C_2-C_1$$

$$-\frac{1}{2}\ln\left|2-y^2\right| = \ln|x| + C_3$$

$$\ln\left|2-y^2\right| = -2\ln|x| - 2C_3$$

$$\ln\left|2-y^2\right| = \ln x^{-2} + C_4$$

Solving for y^2,

$$e^{\ln\left|2-y^2\right|} = e^{\ln x^{-2} + C_4}$$

$$2 - y^2 = e^{\ln x^{-2}} \cdot e^{C_4}$$

$$2 - y^2 = C_5 \cdot x^{-2}$$

$$y^2 = 2 - C_5 x^{-2}$$

So, $y^2 = 2 + Cx^{-2}$.

第四題

$$\frac{dx}{dt} = \frac{\sin\sqrt{t}}{\sqrt{t}};\ x(0) = -1$$

$x(t) = \int \frac{dx}{dt}dt$, so

$$x(t) = \int \frac{\sin\sqrt{t}}{\sqrt{t}}dt$$

Using substitution with $u = \sqrt{t}$ and

$$2du = \frac{1}{\sqrt{t}}dt\ ,$$

$$= 2\int \sin u\, du = 2(-\cos u) + C$$

$$= -2\cos\sqrt{t} + C$$

Since $x(0) = -1$,

$$-1 = -2\cos 0 + C = -2 \cdot 1 + C$$

or $C = 1$

and

$$x(t) = -2\cos\sqrt{t} + 1$$

第五題

$$f_{av} = \frac{1}{1-(-1)}\int_{-1}^{1}[e^{-x}(4-e^{2x})]dx$$

$$= \frac{1}{2}\int_{-1}^{1}(4e^{-x} - e^{x})dx$$

$$= \frac{1}{2}(-4e^{-x} - e^{x})\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{-3}{e} + 3e\right)$$

$$= \frac{3}{2}\left(e - \frac{1}{e}\right)$$

第六題

$$\int_1^2 [3f(x)+2g(x)]dx$$
$$=3\int_1^2 f(x)dx+2\int_1^2 g(x)dx$$
$$=3\left[\int_{-3}^2 f(x)dx-\int_{-3}^1 f(x)dx\right]$$
$$+2\left[\int_{-3}^2 g(x)dx-\int_{-3}^1 g(x)dx\right]$$
$$=3(5-0)+2(-2-4)$$
$$=3$$

$\int_4^4 g(x)dx = G(4)-G(4)=0,$ where $G(x)$ is the antiderivative of $g(x)$.