榮譽第一

國立東華大學 應用數學系 ______學年度第___學期 考試科目：122 □期中□期末考試試卷

第一頁

學號 41111303 姓名 鄭嘉勛 系所 ______ 班別 ____ 任課教師：______ 共___張

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1. Let $Y = X - \mu$

$\Rightarrow f_Y(y) = f_X(y+\mu)\left|\frac{d\,y+\mu}{dy}\right|$

$= \frac{1}{\sigma}\exp\left(-\frac{y}{\sigma}\right)$, $y \geq 0$

is pdf of $NegE(\frac{1}{\sigma})$

$\Rightarrow Y \sim NegE(\frac{1}{\sigma})$

$\Rightarrow EX = E(Y+\mu) = \sigma + \mu = \bar{X}$

$Var X = Var(Y+\mu) = Var Y = \sigma^2 = S^{*2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})$

$\Rightarrow \hat{\sigma}_{MOME} = S^*$

$\hat{\mu}_{MOME} = \bar{X} - S^*$

$\Rightarrow \hat{\eta}_{MOME} = \widehat{e^{-\mu}\log(\sigma)}_{MLE} = e^{-\hat{\mu}}\log(\hat{\sigma}) = e^{-(\bar{X}-S^*)}\log(S^*)$

(b) Goal : Find UMVUE for μ^2

Try $E(\bar{X}^2)$ know $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$= Var(\bar{X}) + (E(\bar{X}))^2$

$= \frac{\sigma^2}{n} + \mu^2$

Try $E(\frac{(n-1)}{\sigma^2}S^2)$ know $\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$

$E(\frac{n-1}{\sigma^2}S^2) = n-1$

$\Rightarrow E(S^2) = \sigma^2$

$\therefore E(\bar{X}^2 - \frac{S^2}{n})$

$= E(\bar{X}^2) - \frac{1}{n}E(S^2)$

$= \mu^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = \mu^2$

Hence $\bar{X} - \frac{S^2}{n}$ is UMVUE for μ^2 ✓

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Goal : Check CRLB is attained

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3. $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\mu \in \mathbb{R}, \sigma^2 > 0$ unknown , $n \geq 3$

let $\theta = (\mu, \sigma^2)$

(a) $f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)$

$= (\sqrt{2\pi}\,\sigma)^{-n}\exp\left(-\frac{1}{2\sigma^2}\Sigma(x_i^2 - 2\mu x_i + \mu^2)\right)$

$= (\sqrt{2\pi}\,\sigma)^{-n}\exp\left(-\frac{1}{2\sigma^2}\left(\Sigma x_i^2 - 2\mu\Sigma x_i + n\mu^2\right)\right)$

$= (\sqrt{2\pi}\,\sigma)^{-n}e^{-n\mu^2}\exp\left(-\frac{1}{2\sigma^2}\Sigma x_i^2 + \frac{\mu}{\sigma^2}\Sigma x_i\right)\cdot 1$

$= C(\theta)\cdot\exp\left(Q_1(\theta)T_1 + Q_2(\theta)T_2\right)h(x)$

and $\mathcal{X} = \mathbb{R}^n \perp\!\!\!\perp \theta$

$\Rightarrow f(\underset{\sim}{x};\theta) \in$ 2-par. exp. family

$B = \{(Q_1(\theta), Q_2(\theta)) \mid \theta \in \mathbb{R}\times(0,\infty)\}$

$= \{(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}) \mid \mu \in \mathbb{R}, \sigma^2 > 0\}$

$= (-\infty, 0)\times\mathbb{R}$

$\Rightarrow B^\circ \neq \phi$

$\Rightarrow T' = (T_1, T_2) = (\sum_{i=1}^{n}X_i^2, \sum_{i=1}^{n}X_i)$

is suff. stat. and complete

$\because (\sum_{i=1}^{n}X_i^2, \sum_{i=1}^{n}X_i) \xleftrightarrow{1-1} (\bar{X}, S^2)$

$\therefore T = (\bar{X}, S^2)$ also is suff. stat.

and complete

CRLB for μ^2 is $\frac{(\frac{\partial}{\partial\mu}\mu^2)^2}{I_n(\mu)}$

$I(\mu) = E\left(\left(\frac{\partial}{\partial\mu}\log f(X;\mu,\sigma^2)\right)^2\right)$

$= E\left(\left(\frac{\partial}{\partial\mu}\log\left(\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(X-\mu)^2}{2\sigma^2}}\right)\right)^2\right)$

$= E\left(\left(\frac{\partial}{\partial\mu}\left(\log\frac{1}{\sqrt{2\pi}\,\sigma} - \frac{1}{2\sigma^2}(X-\mu)^2\right)\right)^2\right)$

$= E\left(\left(-\frac{1}{2\sigma^2}(X-\mu)\cdot 2\cdot(-1)\right)^2\right)$

$= \frac{1}{\sigma^4}E\left((X-\mu)^2\right) = Var X$

$= \frac{1}{\sigma^4}\cdot\sigma^2 = \frac{1}{\sigma^2}$

CRLB $= \frac{(2\mu)^2}{n\cdot\frac{1}{\sigma^2}} = \frac{4}{n}\mu^2\sigma^2$ ✓

$Var(\bar{X}^2 - \frac{1}{n}S^2)$ $\because \bar{X} \perp S^2$ Basu's thm

$= Var(\bar{X}^2) + \frac{1}{n^2}Var(S^2)$ ✓

$Var(\bar{X}$

this is easy

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5. (a) $X_1, \cdots, X_n \overset{iid}{\sim} w/ f(x,\theta) = \theta e^{-\theta x}$, $x>0$, $\theta \in (0,\infty)$, $n \ge 3$

$f(\underline{x};\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$

$= \theta^n e^{-\theta \sum x_i}$

$= \theta^n \exp(-\theta \sum x_i) \cdot 1$

$= C(\theta) \exp(Q(\theta) T) \cdot h(\underline{x})$

$w/ \mathcal{X} = \{\underline{x} \mid x_i > 0\} \perp\!\!\!\perp \theta$

$\Rightarrow f(\underline{x};\theta) \in$ 1-par. exp. family

$\because B = \{Q(\theta) \mid \theta \in (0,\infty)\}$

$= (-\infty, 0)$

$\Rightarrow B^\circ \neq \emptyset$

$\Rightarrow T = \sum_{i=1}^{n} X_i$ is suff. stat. and complete

Try $E(T) = E(\sum X_i) = n \frac{1}{\theta}$

know $T = \sum_{i=1}^{n} X_i \sim Gamma(n, \frac{1}{\theta})$ $\quad \because X_i \overset{iid}{\sim} Gamma(1, \frac{1}{\theta})$

Try $E(\frac{1}{T}) = \int_0^\infty t^{-1} \frac{1}{\Gamma(n)(\frac{1}{\theta})^n} t^{n-1} \exp(-\theta t)\, dt$

$= \underbrace{\int_0^\infty \frac{1}{\Gamma(n-1)(\frac{1}{\theta})^{n-1}} t^{(n-1)-1} \exp(-\theta t)\, dt}_{\text{pdf of } Gamma(n-1, \frac{1}{\theta})} \cdot \frac{\Gamma(n-1)}{\Gamma(n)} \theta$

$= \frac{\theta}{n-1}$

$\therefore E(\frac{n-1}{T}) = \theta$

$\Rightarrow \frac{n-1}{T} = \frac{n-1}{\sum X_i}$ is UMVUE for θ

Goal: Check CRLB for θ is attained

$I(\theta) = E((\frac{\partial}{\partial\theta} \log f(X;\theta))^2)$

$= E((\frac{\partial}{\partial\theta}(\log\theta - \theta X))^2)$

$= E((\frac{1}{\theta} - X)^2)$

$= Var X = \frac{1}{\theta^2}$

$CRLB = \frac{(\frac{\partial}{\partial\theta}\theta)^2}{I_n(\theta)} = \frac{\theta^2}{n}$

$Var(\frac{n-1}{T}) = (n-1)^2 Var(\frac{1}{T})$

where $Var(\frac{1}{T}) = (E(\frac{1}{T})^2) - (E\frac{1}{T})^2 = (\frac{\theta}{n-1})^2$

where $E((\frac{1}{T})^2) = \int_0^\infty t^{-2} \frac{1}{\Gamma(n)(\frac{1}{\theta})^n} t^{n-1} e^{-\theta t}\, dt$

$= \frac{\Gamma(n-2)}{\Gamma(n)} \theta^2 \underbrace{\int_0^\infty \frac{1}{\Gamma(n-2)(\frac{1}{\theta})^{n-2}} t^{(n-2)-1} e^{-\theta t}\, dt}_{\text{pdf of } Gamma(n-2, \frac{1}{\theta})}$

$= \frac{\theta^2}{(n-1)(n-2)}$

$= (n-1)^2 \left(\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2}\right)$

$= \theta^2 (\frac{n-1}{n-2} - 1) = \theta^2 (\frac{1}{n-2}) > \frac{\theta^2}{n}$

$\therefore$ CRLB for θ is not attained

5 (b) $L(\theta \mid \underline{x}) = f(\underline{x};\theta)$ 10

$= \theta^n e^{-\theta \sum x_i}$

$\log L(\theta \mid \underline{x}) = n \log\theta - \theta \sum x_i$

$\frac{\partial}{\partial\theta} \log L(\theta \mid \underline{x}) = \frac{n}{\theta} - \sum x_i \quad > 0$ if $\frac{n}{\theta} > \sum x_i \Leftrightarrow \theta < \frac{n}{\sum x_i}$

$= 0$ if $\theta = \frac{n}{\sum x_i}$

< 0 if $\theta > \frac{n}{\sum x_i}$

$\therefore \hat{\theta}_{MLE} = \frac{n}{\sum X_i}$

$\Rightarrow \widehat{\eta(\theta)}_{MLE} = \widehat{f(c;\theta)}_{MLE} = \widehat{\theta e^{-\theta c}}_{MLE} = \hat{\theta} e^{-\hat{\theta} c}$

$= \frac{n}{\sum X_i} e^{-\frac{n \cdot c}{\sum X_i}}$

6. $n \ge 3$, $X_i \overset{indep}{\sim} P(c_i\theta)$, $c_i > 0$ known $\quad f(x_i;\theta) = e^{-c_i\theta} \frac{(c_i\theta)^x}{x!}$

(a) $I_n(\theta) \overset{indep}{=} I_{X_1}(\theta) + \cdots + I_{X_n}(\theta)$ ✓ 10

which $I_{X_i}(\theta) = E((\frac{\partial}{\partial\theta} \log f(X_i; c_i\theta))^2)$

$= E((\frac{\partial}{\partial\theta}(-c_i\theta + X \log c_i\theta - \log X!))^2)$

$= E((-c_i + c_i \frac{X}{c_i\theta})^2)$

$= \frac{1}{\theta^2} E((X - c_i\theta)^2)$

$= \frac{1}{\theta^2} Var(X)$

$= \frac{c_i}{\theta}$

$\therefore CRLB = \frac{(\frac{\partial}{\partial\theta}(\theta))^2}{I_n(\theta)} = \frac{1^2}{(\frac{c_1}{\theta} + \cdots + \frac{c_n}{\theta})} = \frac{\theta}{\sum c_i}$ ✓

(b) $L(\theta \mid \underline{x}) = f(\underline{x};\theta)$ 10

$= \prod_{i=1}^{n} e^{-c_i\theta} \frac{(c_i\theta)^{x_i}}{x_i!}$

$= e^{-\theta \sum c_i} \prod_{i=1}^{n} (c_i\theta)^{x_i} \prod_{i=1}^{n} \frac{1}{x_i!}$

$\log L(\theta \mid \underline{x}) = -\theta \sum c_i + \sum_{i=1}^{n} x_i \ln c_i\theta + \sum_{i=1}^{n} (-\ln x_i)$

$\frac{\partial}{\partial\theta} \log L(\theta \mid \underline{x}) = -\sum c_i + \sum_{i=1}^{n} \frac{x_i}{c_i\theta} c_i$

$= -\sum c_i + \sum_{i=1}^{n} \frac{x_i}{\theta}$

$= -\sum c_i + \frac{1}{\theta} \sum_{i=1}^{n} x_i \quad > 0$ if $\frac{1}{\theta} \sum x_i > \sum c_i \Leftrightarrow \theta < \frac{\sum x_i}{\sum c_i}$

$= 0$ if $\theta = \frac{\sum x_i}{\sum c_i}$

< 0 if $\theta > \frac{\sum x_i}{\sum c_i}$

$\therefore \hat{\theta}_{MLE} = \frac{\sum X_i}{\sum c_i}$ ✓

$E(\frac{\sum X_i}{\sum c_i}) = \frac{1}{\sum c_i} \sum E(X_i) = \frac{1}{\sum c_i} \sum c_i\theta = \theta \frac{\sum c_i}{\sum c_i} = \theta$

$\therefore \frac{\sum X_i}{\sum c_i}$ is unbias est.

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第三頁

學號 41111303 姓名 鄭嘉勛 系所 ______ 班別 ____ 任課教師：______ 共 __ 張

Continue 6(b)

$Var\left(\frac{\sum X_i}{\sum c_i}\right) \stackrel{\perp}{=} \frac{1}{(\sum c_i)^2}\sum_{i=1}^{n} Var X_i$

$= \frac{1}{(\sum c_i)^2}\sum c_i \theta$

$= \frac{\theta}{\sum c_i} \stackrel{by (a)}{=} CRLB$

$\Rightarrow \frac{\sum X_i}{\sum c_i}$ is UMVUE for θ ✓

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7. $X_1, \cdots, X_n \stackrel{iid}{\sim} w/\ f(x;\theta) = \theta x^{\theta-1}$ $\quad x \in (0,1)$, $n \geq 3$, $\theta \in (0,\infty)$

$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$= \theta^n (x_1 \cdots x_n)^{\theta-1}$

$= \theta^n \exp\left((\theta-1)\ln(x_1\cdots x_n)\right)\cdot 1$

$= c(\theta)\exp\left(Q_1(\theta)\ T\right)\cdot h(x)$

$w/\ \mathcal{X} = \{\underset{\sim}{x} \mid x_i \in (0,1)\}$

$\Rightarrow f(\underset{\sim}{x};\theta) \in$ 1-par. exp. family

$B = \{Q_1(\theta) \mid \theta > 0\}$

$= (-1,\infty)$

$\Rightarrow B^\circ \neq \phi$

$\Rightarrow T = \ln(X_1 \cdots X_n) = \sum \ln X_i$

is suff. stat. and complete.

Goal: find $\delta(T)$ s.t. $E(\delta(T)) = \theta$

$\forall i = 1, \cdots, n$

let $Y_i = -\ln X_i \Rightarrow e^{-Y_i} = X_i$ $\quad x \in (0,1)$ $\quad \Rightarrow -\ln x = y \in (0,\infty)$

$f_{Y_i}(y) = f_{X_i}(e^{-y})\left|\frac{dx(y)}{dy}\right|$

$= \theta (e^{-y})^{\theta-1}\left|\frac{de^{-y}}{dy}\right|$

$= \theta e^{-y(\theta-1)}\cdot|-e^{-y}|$

$= \theta e^{-y\theta}$, $\quad y \in (0,\infty)$

i.e. $Y_i \sim NegE(\theta)$

$\because X_i \stackrel{iid}{}$

$\Rightarrow Y_i \stackrel{iid}{}$

and by 5.

we know $\left[\text{if } X_i \stackrel{iid}{\sim} NegE(\theta),\ \text{then } \frac{n-1}{\sum x_i} \text{ is UMVUE for } \theta\right]$

so, 在這題裡

$\frac{n-1}{\sum Y_i} = \frac{n-1}{\sum(-\ln X_i)} = \frac{n-1}{-\ln(X_1\cdots X_n)} = -\frac{n-1}{T}$

is UMVUE for θ

4. $X_1, \cdots, X_n \stackrel{iid}{\sim} B(1,p)$ $\quad p \in [0,1]$ $\quad n \geq 6$

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$f(\underset{\sim}{x};p) = \prod p^{x_i}(1-p)^{1-x_i}$

$= p^{\sum x_i}(1-p)^{n-\sum x_i}$

$= (1-p)^n \exp\left(\frac{p}{1-p}\sum x_i\right)\cdot 1$

$= c(\theta)\exp\left(Q_1(\theta)\ T\right)\cdot h(\underset{\sim}{x})$

$w/\ \mathcal{X} = \{\underset{\sim}{x} \mid x_i = 0 \text{ or } 1\}$

$\Rightarrow f(\underset{\sim}{x};p) \in$ 1-par. exp. family

$B = \{Q_1(\theta) \mid \theta \in (0,1)\}$

$= \{\frac{p}{1-p} \mid p \in (0,1)\}$

$= (0,\infty)$

$\Rightarrow B^\circ \neq \phi$

$\Rightarrow T = \sum X_i$ is suff. stat. and complete

Try $E(T) = E(\sum X_i) = np$

know $T \sim B(n,p)$

Try $E\left(T(T-1)(n-T)(n-T-1)(n-T-2)\right)$

$= \sum_{t=0}^{n} t(t-1)(n-t)(n-t-1)(n-t-2)\binom{n}{t}p^t(1-p)^{n-t}$

$\because n \geq 6$ $\quad = \sum_{t=2}^{n-3} \cancel{t(t-1)(n-t)(n-t-1)(n-t-2)}\frac{n!}{\cancel{t!(n-t)!}\ (t-2)!(n-t-3)!} p^{t-2}(1-p)^{n-t-3}\ p^2(1-p)^3$

$= \left[\sum_{t=2}^{n-5}\underbrace{\frac{(n-5)!}{(t-2)!(n-t-3)!}p^{t-2}(1-p)^{n-t-3}}_{\text{pdf of } B(n-5,p)}\right]\frac{n!}{(n-5)!}\cdot p^2(1-p)^3$

$= 1\cdot\frac{n!}{(n-5)!}p^2(1-p)^3$

$\therefore \frac{(n-5)!}{n!}T(T-1)(n-T)(n-T-1)(n-T-2)$

is UMVUE for $p^2(1-p)^3$

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2. $X_1, \cdots, X_n \sim N(\theta, 1) \Rightarrow X_i - \theta \sim N(0,1)$

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$Y \triangleq \#$ of $X_1, \cdots, X_n > 0$

so. $Y \sim Bin(n, p)$ w/ $p = 1 - \Phi(0-\theta)$ w/ Φ is cdf. of $N(0,1)$

$= \Phi(\theta)$

Goal : Find MLE of θ

$L(\theta | y) = f_Y(y;\theta)$

$= \binom{n}{y} (\Phi(\theta))^y \underbrace{(1-\Phi(\theta))}_{=\Phi(-\theta)}{}^{n-y}$

$\Rightarrow \frac{\partial}{\partial\theta} \log L(\theta|Y) = \frac{\partial}{\partial\theta}\left(\log\binom{n}{Y} + Y\log(\Phi(\theta)) + (n-Y)\log(\Phi(-\theta))\right)$

$= Y \frac{1}{\Phi(\theta)}\phi(\theta) + (n-Y)\frac{1}{\Phi(-\theta)}(-\phi(\theta))$ w/ ϕ is p.d.f. of $N(0,1)$

$= \phi(\theta)\left[\frac{Y}{\Phi(\theta)} + \frac{Y-n}{1-\Phi(\theta)}\right]$

$= \phi(\theta)\left[\frac{Y - n\Phi(\theta)}{\Phi(\theta)(1-\Phi(\theta))}\right]$

$= \underbrace{\frac{\phi(\theta)}{\Phi(\theta)(1-\Phi(\theta))}}_{>0, \forall\theta} (Y - n\Phi(\theta))$ >0 if $Y > n\Phi(\theta) \Leftrightarrow \Phi^{-1}(\frac{Y}{n}) > \theta$

$=0$ if $\theta = \Phi^{-1}(\frac{Y}{n})$

<0 if $\theta > \Phi^{-1}(\frac{Y}{n})$

$\therefore \hat{\theta}_{MLE} = \Phi^{-1}\left(\frac{Y}{n}\right)$

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