

* Answer and mark clearly the questions in the provided answer sheets.
Write down your name and student's ID on the each answer sheet you used.

* **Note:** There are 120 points in this exam.

No points will be given if no arguments are provided for an answer.

Good Luck!

~~ Yuling ☺

Use $z_{0.025} = 1.96$, $z_{0.05} = 1.645$, $z_{0.1} = 1.28$, $z_{0.125} = 1.15$, $z_{0.2} = .845$, $z_{0.25} = .675$
 $\chi_{1,0.05}^2 = 3.84$, $\chi_{2,0.05}^2 = 5.99$, $\chi_{3,0.05}^2 = 7.8$, $\chi_{4,0.05}^2 = 9.5$, $\chi_{5,0.05}^2 = 11.07$, $\chi_{6,0.05}^2 = 12.59$
 $\chi_{1,0.95}^2 = .004$, $\chi_{2,0.95}^2 = .103$, $\chi_{3,0.95}^2 = .352$, $\chi_{4,0.95}^2 = .711$, $\chi_{5,0.95}^2 = 1.15$, $\chi_{6,0.95}^2 = 1.64$,
 $t_{7,0.05} = 1.9$, $t_{7,0.1} = 1.4$, $t_{8,0.05} = 1.85$, $t_{8,0.1} = 1.39$, $t_{9,0.05} = 1.83$, $t_{9,0.1} = 1.38$
 $F_{1,7,0.05} = 5.6$, $F_{1,7,0.1} = 3.6$, $F_{1,8,0.05} = 5.3$, $F_{1,8,0.1} = 3.45$ in this exam.

- (10 points) State the Neyman-Pearson Fundamental Lemma
- Let $\alpha \in (0, 1)$, $0 < \theta_1 < \theta_0$ be given and X_1, \dots, X_n , $n > 1$, be a random sample from $f(x; \theta)$ where

$$f(x; \theta) = \theta/x^2, \quad \theta \leq x < \infty.$$

- (10 points) Derive the MP (most powerful) level α test for testing

$H_0 : \theta = \theta_0$ v.s. $H_1 : \theta = \theta_1$. Calculate its power.

- (10 points) Derive the UMP (uniformly most powerful) level α test for testing $H_0 : \theta \geq \theta_0$ v.s. $H_1 : \theta < \theta_0$.

Is the UMP the same test as your answer to the previous part of this question?

- (10 points) Show that a UMP level α test, if exists, must be an unbiased test, hence a UMPU (uniformly most powerful unbiased) level α test.
- Let X_1, \dots, X_n be i.i.d. random variables from $N(\theta, \sigma_0^2)$, $\sigma_0^2 > 0$ is known. Given a constant $\theta_0 \in R$, and $\alpha \in (0, 1)$.

- (10 points) Derive a LRT (likelihood ratio test) with level α for testing

$H_0 : \theta \geq \theta_0$ v.s. $H_1 : \theta < \theta_0$.

Is it UMP?

- (10 points) Suppose that it is desired to have level $\alpha = 0.05$ and a maximum Type-II error probability 0.2 at $\theta = \theta_0 - 0.01\sigma$, find the smallest sample size n that will achieve this requirement.

- Let X_1, \dots, X_n , $n > 1$, be i.i.d. r.v.'s with p.d.f. $f(x; \theta) = \theta(1-x)^{\theta-1}$, for $x \in (0, 1)$, and $\theta > 0$ is the unknown parameter. Given $\theta_0 > 0$ and $\alpha \in (0, 1)$.

- (10 points) Derive the UMP level α test for testing $H_0 : \theta \leq \theta_0$ v.s. $H_1 : \theta > \theta_0$.

- (10 points) Describe how one can determine the minimum sample size n required to obtain power at least 0.9 at $\theta_1 = 1000$ when $\theta_0 = 250$ and $\alpha = 0.05$.

- (20 points) Let X_1, \dots, X_n be a random sample from the $U[0, \theta]$, $\theta > 0$; and $x_{(n)} = \max(x_1, \dots, x_n)$. Given a $\theta_0 > 0$ and $\alpha \in (0, 1)$. Show that the test

$$\phi(\mathbf{x}) = \phi(\mathbf{x}_{(n)}) = \begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0 \sqrt[n]{\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

is a UMP level α test for testing $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta \neq \theta_0$.

7. (10 points) Consider a group of 100 people living and working under very similar conditions. Half of them are given a preventive shot against a certain disease and the other half serve as control. Of those received the treatment, 40 did not contract the disease whereas the remaining 10 did so. Of those not treated, 30 did contract the disease and the remaining 20 did not. Test effectiveness of the vaccine at the level of significance $\alpha = 0.05$
8. (10 points) Let X_1, \dots, X_n , $n > 1$, be independent r.v.'s from a population X . With level $\alpha \leq 0.05$, to test $H_0 : X \sim \text{Gamma}(\theta, 1), 1 \leq \theta \leq 6$ v.s. $H_1 : X \sim P(\theta), 1 \leq \theta \leq 6$, is there optimal test here? If so, indicate what the optimal test should be. If not, explain briefly.