$\star$ Answer and mark clearly the questions in the provided answer sheets.
Write down your name and student's ID on the each answer sheet you used.
$*$ **Note:** There are 100 points in this exam.
No points will be given if no arguments are provided for an answer.
Good Luck! $\sim\sim$ *Yuling* ☺

Use $z_{0.025} = 1.96$, $z_{0.05} = 1.645$, $z_{0.1} = 1.28$, $z_{0.125} = 1.15$, $z_{0.2} = .845$, $z_{0.25} = .675$
$\chi^2_{1,0.05} = 3.84$, $\chi^2_{2,0.05} = 5.99$, $\chi^2_{3,0.05} = 7.8$, $\chi^2_{4,0.05} = 9.5$, $\chi^2_{5,0.05} = 11.07$, $\chi^2_{6,0.05} = 12.59$
$\chi^2_{1,0.95} = .004$, $\chi^2_{2,0.95} = .103$, $\chi^2_{3,0.95} = .352$, $\chi^2_{4,0.95} = .711$, $\chi^2_{5,0.95} = 1.15$, $\chi^2_{6,0.95} = 1.64$,
$t_{7,0.05} = 1.9$, $t_{7,0.1} = 1.4$, $t_{8,0.05} = 1.85$, $t_{8,0.1} = 1.39$, $t_{9,0.05} = 1.83$, $t_{9,0.1} = 1.38$
$F_{1,7,0.05} = 5.6$, $F_{1,7,0.1} = 3.6$, $F_{1,8,0.05} = 5.3$, $F_{1,8,0.1} = 3.45$ in this exam.

1. (10 points) A sample of size 1 is taken from the *p.d.f.*

$$f(x;\theta) = \begin{cases} \frac{2}{\theta^2}(\theta - x) & \text{if } 0 < x < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

   Let θ_0, θ_1 with $0 < \theta_0 < \theta_1$ be two given positive constants.
   Find the most powerful (MP) level $\alpha = 0.05$ test for testing
   $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta = \theta_1$. Calculate its power.

2. Let $X \sim f(x;\theta)$ where

$$f(x;\theta) = 2\theta x + 2(1-\theta)(1-x), \quad 0 < x < 1, \quad \theta \in [0,1].$$

   (a) (10 points) Based on X, derive a LRT (likelihood ratio test) level α test, $\alpha \in (0, 0.5)$, for testing $H_0 : \theta \in \{0,\ 0.5\}$ $\quad v.s.\ H_1 : \theta \in \{0.75,\ 1\}$.

   (b) (10 points) Is your answer to part (a) uniformly most powerful (UMP) at level α? Calculate its power function. Is it unbiased?

   (c) (10 points) Is your answer to part (a) also the UMP level α test for testing $H_0 : \theta \le 0.5$ $\quad v.s.\ H_1 : \theta > 0.5$.

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables from $N(\theta, \sigma_0^2)$, σ_0^2 is known. Given a constant $\theta_0 \in R$.

   (a) (10 points) Derive a LRT with level α for testing $H_0 : \theta \ge \theta_0$ $\quad v.s.\ H_1 : \theta < \theta_0$. Is it UMP? And, is it UMPU?

   (b) (10 points) Suppose that it is desired to have level $\alpha = 0.1$ and power at least .95 at $\theta = \theta_0 - 0.5\,\sigma_0$, find the smallest sample size n that will achieve this requirement.

4. Let Let $X_1, \ldots, X_n$, $n > 1$, be a random sample from the p.d.f.
   $f(x;\theta) = \theta(1-x)^{\theta-1}$, for $x \in (0,1)$, $\theta \in \Omega = (0,\infty)$; and $\alpha \in (0,1)$, $\theta_0 > 0$ are given.

   (a) (10 points) Derive the UMP level $\alpha = 0.05$ test of $H_0 : \theta \le \theta_0$ $\quad v.s.\ H_1 : \theta > \theta_0$.

   (b) (10 points) Calculate its power, and indicate how the sample size n can be determined if it is required that, at $\theta_1 = 10\ \theta_0$, power of this UMP test be at least 0.9.

Result A: Let $X_1, \ldots, X_n$ be a random sample from the $U[0, \theta]$, $\theta > 0$. Given a $\theta_0 > 0$ and $\alpha \in (0, 1)$, the test

$$\phi(\mathbf{x}) = \phi(\mathbf{x_{(n)}}) = \begin{cases} 1 & \text{if } x_{(n)} = \max(x_1, \ldots, x_n) > \theta_0 \text{ or } x_{(n)} \leq \theta_0 \sqrt[n]{\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

is a UMP level α test for testing $H_0: \theta = \theta_0$ $v.s.$ $H_1: \theta \neq \theta_0$.

5. (10 points) Let $Y_1, \ldots, Y_n$ are *i.i.d.* r.v.'s from the common p.d.f. $f(y; \mu)$,where $f(y; \mu) = e^{-(y-\mu)}$, for $y \geq \mu$, and $f(y; \mu) = 0$, otherwise; $\mu \in \Omega = R$. Give $\mu_o \in R$ and $\alpha \in (0, 1)$, utilizing **Result A** above to derive the UMP level α test for testing $H_0: \mu = \mu_0$ $v.s.$ $H_1: \mu \neq \mu_0$.

6. (10 points) Let $X_1, \ldots, X_n$, $n > 1$, be independent r.v.'s from a population X.
   With level $\alpha \leq 0.05$, to test
   $H_0: X \sim N(\theta, 1), 1 \leq \theta \leq 6$ $v.s.$ $H_1: X \sim P(\theta), 1 \leq \theta \leq 6$,
   is there optimal test here ?
   If so, indicate what the optimal test should be.
   If not, explain briefly.