⋆ Answer and mark clearly the questions in the provided answer sheets.
Write down your name and student's ID on the each answer sheet you used.
∗ **Note:** There are 125 points in this exam.
No points will be given if no arguments are provided for an answer.
Good Luck and happy winter break ! ∼∼ *Yes* ⌣̈

1. (10 points) Let $X_1, \dots, X_n$, $n \geq 3$, be *i.i.d* random variables with p.d.f. $f(\cdot;\mu,\sigma)$ given by
$$f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right), \quad x \geq \mu \quad \theta = (\mu,\sigma) \in \Omega = R \times (0,\infty).$$
Find a MOME (method of moments estimator) for $\eta(\theta) = e^{-\mu}\log(\sigma)$.

2. (10 points) Suppose that n, $n \geq 3$, random observations were taken from the population $X \sim N(\theta, 1)$ in a study from which a research paper was published. Usually, the values of the original observations will not be given in a published paper. If, say, from that paper you only know Y : the number among those n observations which is greater than 0.
Base on Y, find the MLE (maximum likelihood estimator) of θ.

3. $X_1, \dots, X_n$, $n \geq 3$, be an *i.i.d* sample from $N(\mu, \sigma^2)$, both $\mu \in R$ and $\sigma > 0$ are unknown.
Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$, where $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.
   (a) (10 points) Show that the statistic $T(X) = (\bar{X}, S^2)$ is sufficient for $\theta = (\mu, \sigma^2)$ and is complete.
   (b) (20 points) Find the UMVUE (uniformly minimum variance unbiased estimator) for μ^2. Investigate if the Cramér-Rao bound is attained.

4. (10 points) Let $X_1, \dots, X_n$, $n \geq 6$, be an *i.i.d* sample from $B(1,p)$, $p \in [0,1]$.
Find the UMVUE of $p^2(1-p)^3$.

5. Let $X_1, \dots, X_n$, $n \geq 3$, be *i.i.d.* random variables with *p.d.f.*
$f(x;\theta) = \theta\exp(-\theta x)$, $x > 0$, $\theta \in \Omega = (0,\infty)$.
   (a) (20 points) Find the UMVUE of θ. Investigate if the Cramér-Rao bound is attained.
   (b) (10 points) Let $\eta(\theta) = f(c;\theta)$ and $c > 0$ a given constant.
   Find the MLE of $\eta(\theta)$.

6. Suppose that, $n \geq 3$,for $i = 1, \dots, n$, $X_i \sim \mathrm{P}(c_i\theta)$, where $c_i > 0$, $i = 1, 2, \dots, n$, are given positive constants, and assume that $X_1, X_2, \dots, X_n$ are independent.
   (a) (10 points) Calculate the Cramér-Rao lower bound when estimating θ.
   (b) (10 points) Find the MLE of θ, and prove (or disprove) it is the UMVUE of θ.

7. (15 points) Let $X_1, \dots, X_n$, $n \geq 3$, be *i.i.d.* random variables with *p.d.f.*
$f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta \in \Omega = (0,\infty)$.
Find the UMVUE of θ