8.6 L'Hôpital's Rule

- Approximate limits that produce indeterminate forms.
- Use L'Hôpital's Rule to evaluate limits.

Indeterminate Forms and Limits

In Sections 1.5 and 3.6, you studied limits such as

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

and

$$\lim_{x \to \infty} \frac{2x + 1}{x + 1}.$$

In those sections, you discovered that direct substitution can produce an **indeterminate form** such as $0/0$ or ∞/∞. For instance, if you substitute $x = 1$ into the first limit, you obtain

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0} \qquad \text{Indeterminate form}$$

which tells you nothing about the limit. To find the limit, you can factor and divide out like factors, as shown.

$$\begin{aligned}
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} && \text{Factor.} \\
&= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}} && \text{Divide out like factors.} \\
&= \lim_{x \to 1} (x + 1) && \text{Simplify.} \\
&= 1 + 1 && \text{Direct substitution} \\
&= 2 && \text{Simplify.}
\end{aligned}$$

For the second limit, direct substitution produces the indeterminate form ∞/∞, which again tells you nothing about the limit. To evaluate this limit, you can divide the numerator and denominator by x. Then you can use the fact that the limit of $1/x$, as $x \to \infty$, is 0.

$$\begin{aligned}
\lim_{x \to \infty} \frac{2x + 1}{x + 1} &= \lim_{x \to \infty} \frac{2 + (1/x)}{1 + (1/x)} && \text{Divide numerator and denominator by } x. \\
&= \frac{2 + 0}{1 + 0} && \text{Evaluate limits.} \\
&= 2 && \text{Simplify.}
\end{aligned}$$

Algebraic techniques such as these tend to work well as long as the function itself is algebraic. To find the limits of other types of functions, such as exponential functions or trigonometric functions, you generally need to use a different approach.

Example 1 | Approximating a Limit

Find the limit.

$$\lim_{x\to 0} \frac{e^{3x}-1}{x}$$

SOLUTION When evaluating a limit, you can choose from three basic approaches. That is, you can attempt to find the limit *analytically*, *graphically*, or *numerically*. For this particular limit, it is not clear how to use an analytic approach because direct substitution yields an indeterminate form.

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = \lim_{x\to 0} \frac{e^{3(0)}-1}{0} \qquad \text{Indeterminate form}$$

$$= \frac{0}{0} \qquad \text{Indeterminate form}$$

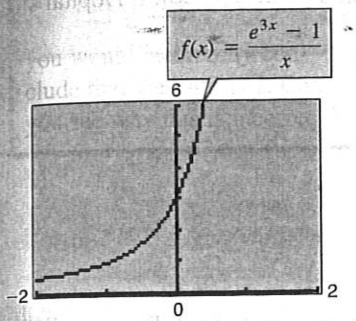

FIGURE 8.38

Using a graphical approach, you can graph the function, as shown in Figure 8.38, and then use the *zoom* and *trace* features to estimate the limit. Using a numerical approach, you can construct a table, such as that shown below.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$\frac{e^{3x}-1}{x}$	2.9554	2.9955	2.9996	?	3.0005	3.0045	3.0455

From the values in the table, it appears that the limit is 3. So, from either a graphical or a numerical approach, you can approximate the limit to be

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = 3.$$

Checkpoints 1

Find the limit.

$$\lim_{x\to 0} \frac{1-e^{2x}}{x}$$

Example 2 | Approximating a Limit

Find the limit.

$$\lim_{x\to 0} \frac{\sin 4x}{x}$$

SOLUTION As in Example 1, it is not clear how to use an analytic approach because direct substitution yields an indeterminate form.

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} \frac{\sin 4(0)}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

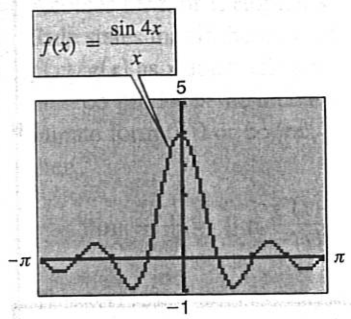

FIGURE 8.39

Using a graphical approach, you can graph the function, as shown in Figure 8.39, and then, using the *zoom* and *trace* features, you can estimate the limit to be 4. A numerical approach would lead to the same conclusion. So, using either a graphical or a numerical approach, you can approximate the limit to be

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$$

Checkpoints 2

Find the limit.

$$\lim_{x\to 0} \frac{\sin(x/2)}{x}$$

L'Hôpital's Rule

L'Hôpital's Rule, which is named after the French mathematician Guillaume Francois Antoine de L'Hôpital (1661–1704), describes an analytic approach for evaluating limits.

L'Hôpital's Rule

Let (a, b) be an interval that contains c. Let f and g be differentiable in (a, b), except possibly at c. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists or is infinite. The indeterminate form ∞/∞ comes in four forms: ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, and $(-\infty)/(-\infty)$. L'Hôpital's Rule can be applied to each of these forms.

Example 3 | Using L'Hôpital's Rule

Find the limit.

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x}$$

SOLUTION In Example 1, it was shown that the limit appears to be 3. Because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain the same result.

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}[e^{3x} - 1]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{3e^{3x}}{1}$$ Differentiate numerator and denominator separately.

$$= \frac{3e^{3(0)}}{1}$$ Direct substitution

$$= 3$$ Simplify.

Checkpoint 3

Find the limit using L'Hôpital's Rule.

$$\lim_{x \to 0} \frac{1 - e^{2x}}{x}$$

Study Tip

Be sure you see that L'Hôpital's Rule involves $f'(x)/g'(x)$, *not* the derivative of the quotient $f(x)/g(x)$.

Example 4 | Using L'Hôpital's Rule

Find the limit.

$$\lim_{x\to 0} \frac{\sin 4x}{x}$$

Study Tip

You *cannot* apply L'Hôpital's Rule unless direct substitution produces the indeterminate form $0/0$ or ∞/∞. For instance, if you were to apply L'Hôpital's Rule to the limit

$$\lim_{x\to 0} \frac{x}{e^x}$$

you would incorrectly conclude that the limit is 1. Can you see why this is incorrect?

SOLUTION In Example 2, it was shown that the limit appears to be 4. Because direct substitution produces the indeterminate form $0/0$

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \frac{0}{0} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule to obtain the same result.

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin 4x]}{\frac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to 0} \frac{4\cos 4x}{1} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= \frac{4\cos[4(0)]}{1} \qquad \text{Direct substitution}$$

$$= 4 \qquad \text{Simplify.}$$

Checkpoint 4

Find the limit using L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\sin(x/2)}{x}$$

Example 5 | Using L'Hôpital's Rule

Find the limit.

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1}$$

Study Tip

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ or $-\infty$ produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

This form is used in Example 5.

SOLUTION Because direct substitution produces the indeterminate form ∞/∞

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} = \frac{\infty}{\infty} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule, as shown.

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} = \lim_{x\to\infty} \frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[e^{2x}+1]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to\infty} \frac{e^x}{2e^{2x}} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= \lim_{x\to\infty} \frac{1}{2e^x} \qquad \text{Simplify.}$$

$$= 0 \qquad \text{Evaluate limit.}$$

Checkpoint 5

Find the limit.

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{4x}+1}$$

Sometimes it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form. This is shown in Example 6.

Example 6 | Using L'Hôpital's Rule Repeatedly

Find the limit.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$$

SOLUTION Because direct substitution results in the indeterminate form ∞/∞

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{\infty}{\infty} \qquad \text{Indeterminate form}$$

you can apply L'Hôpital's Rule, as shown.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= \frac{-\infty}{-\infty} \qquad \text{Indeterminate form}$$

After one application of L'Hôpital's Rule, you still obtain an indeterminate form. In such cases, you can try L'Hôpital's Rule again, as shown.

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to-\infty} \frac{2}{e^{-x}} \qquad \text{Differentiate numerator and denominator separately.}$$

$$= 0 \qquad \text{Evaluate limit.}$$

So, you can conclude that the limit is zero.

Checkpoint 6

Find the limit.

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}}$$

Study Tip

Remember that even in those cases in which L'Hôpital's Rule can be applied to determine a limit, it is still a good idea to confirm the result graphically or numerically. For instance, the table below provides a numerical confirmation of the limit in Example 6.

x	-2	-4	-6	-8	-10	-12
$\frac{x^2}{e^{-x}}$	0.5413	0.2931	0.0892	0.0215	0.0045	0.0008

L'Hôpital's Rule can be used to compare the rates of growth of two functions. For instance, consider the limit in Example 5

$$\lim_{x\to\infty} \frac{e^x}{e^{2x}+1} = 0.$$

Both of the functions $f(x) = e^x$ and $g(x) = e^{2x} + 1$ approach infinity as $x \to \infty$. However, because the quotient $f(x)/g(x)$ approaches 0 as $x \to \infty$, it follows that the denominator is growing much more rapidly than the numerator.

Study Tip

L'Hôpital's Rule is necessary to solve certain real-life problems such as compound interest problems and other business applications.

Example 7 | Comparing Rates of Growth

Each of the functions below approaches infinity as x approaches infinity. Which function has the highest rate of growth?

a. $f(x) = x$ **b.** $g(x) = e^x$ **c.** $h(x) = \ln x$

SOLUTION Using L'Hôpital's Rule, you can show that each of the limits is zero.

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[e^x]} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[e^x]} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

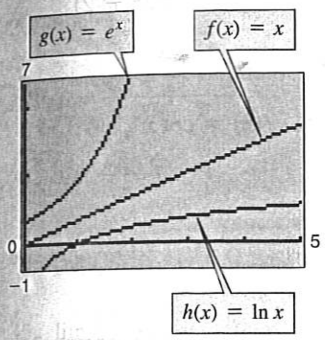

FIGURE 8.40

From this, you can conclude that $h(x) = \ln x$ has the lowest rate of growth, $f(x) = x$ has a higher rate of growth, and $g(x) = e^x$ has the highest rate of growth. This conclusion is confirmed graphically in Figure 8.40. ■

Checkpoint 7

Use a graphing utility to order the functions according to the rate of growth of each function as x approaches infinity.

a. $f(x) = e^{2x}$ **b.** $g(x) = x^2$ **c.** $h(x) = \ln x^2$ ■

SUMMARIZE
Section 8.6

1. State L'Hôpital's Rule *(page 564)*. For an example of using L'Hôpital's Rule to evaluate indeterminate form, see Examples 3, 4 and 5.
2. Describe a real-life example of how L'Hôpital's Rule can be used to compare rates of growth *(page 516, Example 7)*.

SKILLS WARM UP 8.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, find the limit, if it exists.

1. $\lim_{x\to\infty} (x^2 - 10x + 1)$
2. $\lim_{x\to\infty} \frac{2x + 5}{3}$
3. $\lim_{x\to\infty} \frac{x + 1}{x^2}$
4. $\lim_{x\to\infty} \frac{5x - 2}{x^2 + 1}$
5. $\lim_{x\to\infty} \frac{2x^2 - 5x + 7}{3x^2 + 12x + 4}$
6. $\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 4}$
7. $\lim_{x\to\infty} \frac{x^2 - 5}{2x + 7}$
8. $\lim_{x\to\infty} \frac{x^2 - 2x + 1}{x - 4}$

In Exercises 9–12, find the first derivative of the function.

9. $f(x) = \cos x^2$
10. $f(x) = \sin(5x - 1)$
11. $f(x) = \sec 4x$
12. $f(x) = \tan(x^2 - 2)$

In Exercises 13–16, find the second derivative of the function.

13. $f(x) = \sin(2x + 3)$
14. $f(x) = \cos \frac{x}{2}$
15. $f(x) = \tan x$
16. $f(x) = \cot x$

Exercises 8.6

In Exercises 1– 6, decide whether the limit produces an indeterminate form.

1. $\lim_{x\to 0} \frac{2x + \sqrt{x}}{x}$
2. $\lim_{x\to\infty} \frac{x^2 + 4x - 3}{7x^2 + 2}$
3. $\lim_{x\to -\infty} \frac{4}{x^2 + e^x}$
4. $\lim_{x\to 0} \frac{\sin x}{e^x}$
5. $\lim_{x\to\infty} \frac{2xe^{2x}}{3e^x}$
6. $\lim_{x\to\infty} \frac{\ln x}{x}$

In Exercises 7–10, complete the table to estimate the limit numerically.

7. $\lim_{x\to 0} \frac{e^{-x} - 1}{3x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

8. $\lim_{x\to 3} \frac{x^2 + x - 12}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

9. $\lim_{x\to 0} \frac{\sin x}{5x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

10. $\lim_{x\to 0} \frac{\cos 2x - 1}{6x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

In Exercises 11–16, use a graphing utility to find the indicated limit graphically.

11. $\lim_{x\to\infty} \frac{\ln x}{x^2}$
12. $\lim_{x\to\infty} \frac{e^x}{x^2}$
13. $\lim_{x\to 1} \frac{\ln(2 - x)}{x - 1}$
14. $\lim_{x\to 1} \frac{e^{x-1}}{x^2 - 1}$
15. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 5x + 6}$
16. $\lim_{x\to 1} \frac{x^2 + 3x - 4}{x^2 + 2x - 3}$

In Exercises 17–48, use L'Hôpital's Rule to find the limit. You may need to use L'Hôpital's Rule repeatedly.

17. $\lim_{x\to 0} \dfrac{e^{-x}-1}{x}$

18. $\lim_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$

19. $\lim_{x\to 0} \dfrac{\sin x}{5x}$

20. $\lim_{x\to 0} \dfrac{\cos 2x - 1}{6x}$

21. $\lim_{x\to \infty} \dfrac{\ln x}{x^2}$

22. $\lim_{x\to \infty} \dfrac{e^x}{x^2}$

23. $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2-5x+6}$

24. $\lim_{x\to 1} \dfrac{x^2+3x-4}{x^2+2x-3}$

25. $\lim_{x\to 0} \dfrac{2x+1-e^x}{x}$

26. $\lim_{x\to 0} \dfrac{2x-1+e^{-x}}{3x}$

27. $\lim_{x\to \infty} \dfrac{\ln x}{e^x}$

28. $\lim_{x\to \infty} \dfrac{3x}{e^x}$

29. $\lim_{x\to \infty} \dfrac{4x^2+2x-1}{3x^2-7}$

30. $\lim_{x\to \infty} \dfrac{3x^2+5}{2x^2-11}$

31. $\lim_{x\to \infty} \dfrac{1-x}{e^x}$

32. $\lim_{x\to 0} \dfrac{e^{3x}-1}{2x}$

33. $\lim_{x\to 1} \dfrac{\ln x}{x^2-1}$

34. $\lim_{x\to 2} \dfrac{\ln(x-1)}{x-2}$

35. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 5x}$

36. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

37. $\lim_{x\to \pi/2} \dfrac{2\cos x}{3\cos x}$

38. $\lim_{x\to 0} \dfrac{\cos 2x - 1}{x}$

39. $\lim_{x\to 0} \dfrac{\sin x}{e^x-1}$

40. $\lim_{x\to 1} \dfrac{\ln x}{1-x^2}$

41. $\lim_{x\to \pi} \dfrac{\cos(x/2)}{x-\pi}$

42. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

43. $\lim_{x\to \infty} \dfrac{x}{\sqrt{x+1}}$

44. $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

45. $\lim_{x\to 0} \dfrac{\sqrt{4-x^2}-2}{x}$

46. $\lim_{x\to \infty} \dfrac{x^2+2}{e^{x^2}}$

47. $\lim_{x\to \infty} \dfrac{e^{3x}}{x^3}$

48. $\lim_{x\to \infty} \dfrac{x^2}{e^x \ln x}$

In Exercises 49–56, find the limit. (*Hint:* L'Hôpital's Rule does not apply in every case.)

49. $\lim_{x\to \infty} \dfrac{x^2+2x+1}{x^2+3}$

50. $\lim_{x\to \infty} \dfrac{3x^3-7x+5}{4x^3+2x-11}$

51. $\lim_{x\to -1} \dfrac{x^3+3x^2-6x-8}{2x^3-3x^2-5x+6}$

52. $\lim_{x\to 0} \dfrac{2x^3-x^2-3x}{x^4+2x^3-9x^2-18x}$

53. $\lim_{x\to 3} \dfrac{\ln(x-2)}{x-2}$

54. $\lim_{x\to -1} \dfrac{\ln(x+2)}{x+2}$

55. $\lim_{x\to 1} \dfrac{2\ln x}{e^x}$

56. $\lim_{x\to 0} \dfrac{\sin \pi x}{x}$

In Exercises 57–62, use L'Hôpital's Rule to compare the rates of growth of the numerator and the denominator.

57. $\lim_{x\to \infty} \dfrac{x^2}{e^{4x}}$

58. $\lim_{x\to \infty} \dfrac{x^3}{e^{2x}}$

59. $\lim_{x\to \infty} \dfrac{(\ln x)^4}{x}$

60. $\lim_{x\to \infty} \dfrac{(\ln x)^2}{x^4}$

61. $\lim_{x\to \infty} \dfrac{(\ln x)^n}{x^m}$

62. $\lim_{x\to \infty} \dfrac{x^m}{e^{nx}}$

63. Complete the table to show that x eventually "overpowers" $(\ln x)^5$.

x	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{(\ln x)^5}{x}$						

64. Complete the table to show that e^x eventually "overpowers" x^6.

x	1	2	4	8	12	20	30
$\dfrac{e^x}{x^6}$							

In Exercises 65–68, L'Hôpital's Rule is used incorrectly. Describe the error.

65. $\lim_{x\to 0} \dfrac{e^{3x}-1}{e^x} = \lim_{x\to 0} \dfrac{3e^{3x}}{e^x} = \lim_{x\to 0} 3e^{2x} = 3$

66. $\lim_{x\to 0} \dfrac{\sin \pi x + 1}{x} = \lim_{x\to 0} \dfrac{\pi \cos \pi x}{1} = \pi$

67. $\lim_{x\to 1} \dfrac{e^x-1}{\ln x} = \lim_{x\to 1} \dfrac{e^x}{(1/x)} = \lim_{x\to 1} xe^x = e$

68. $\lim_{x\to \infty} \dfrac{e^{-x}}{1-e^{-x}} = \lim_{x\to \infty} \dfrac{-e^{-x}}{e^{-x}} = \lim_{x\to \infty} -1 = -1$