

1. Let  $X_1, \dots, X_n$  be a random sample from a  $U(0, \theta)$ , with unknown parameter  $\theta > 0$ , and let  $Y = X_{(n)} = \max\{X_1, \dots, X_n\}$ .
  - (a) Let  $0 \leq c < d$  be given constant, and consider the random interval  $C_1(Y) = [Y + c, Y + d]$ . Calculate the coverage probability function of  $C_1(Y)$ , then show that the confidence coefficient of  $C_1(Y)$  is 0.
  - (b) Let  $1 \leq a < b$  be given constant, and consider the random interval  $C_2(Y) = [aY, bY]$ . Show that the coverage probability function of  $C_2(Y)$  is a constant, hence equals to the confidence coefficient of  $C_2(Y)$ .
  - (c) Now, for a given  $\alpha \in (0, 1)$ , give three different pairs of  $a$  and  $b$  in (b) such that the confidence coefficient of  $C_2(Y)$  is  $1 - \alpha$ .
2. Assume that  $\alpha_1$  and  $\alpha_2$  are any numbers such that  $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ . Let  $U_\alpha$  be an exact upper confidence bound for  $\theta$ , i.e.  $P_\theta(\theta \leq U_\alpha(X)) = \alpha, \forall \theta$ . If  $U_{\alpha_1}(x) \leq U_{\alpha_2}(x), \forall x$ , prove that for all  $\theta, P_\theta\{\theta \in [U_{\alpha_1}, U_{\alpha_2}]\} \geq \alpha_2 - \alpha_1$ .
3. Let  $X_1, \dots, X_n$  be *i.i.d.* from  $N(\theta, \sigma_0^2)$ , with  $\sigma_0 > 0$  and  $\alpha \in (0, 1)$  are given constants; also let  $X = (X_1, \dots, X_n)$ .
  - (a) Consider the random lower confidence limit for  $\theta$ :

$$C_1(X) = \left[ \bar{X} - \frac{\sigma_0}{\sqrt{n}} Z_\alpha, \infty \right).$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of  $C_1(X)$ . Is  $C_1(X)$  an unbiased confidence set for  $\theta$ ?

- (b) Also, consider the random upper confidence limit for  $\theta$ :

$$C_2(X) = \left( -\infty, \bar{X} + \frac{\sigma_0}{\sqrt{n}} Z_\alpha \right].$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of  $C_2(X)$ . Is  $C_2(X)$  an unbiased confidence set for  $\theta$ ?

- (c) Now, consider the random confidence interval for  $\theta$ :

$$C_3(X) = \left[ \bar{X} - \frac{\sigma_0}{\sqrt{n}} Z_{\alpha/2}, \bar{X} + \frac{\sigma_0}{\sqrt{n}} Z_{\alpha/2} \right].$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of  $C_3(X)$ . Is  $C_3(X)$  an unbiased confidence set for  $\theta$ ?

4. Find a  $1 - \alpha$  confidence interval for  $\theta$ , given  $X_1, \dots, X_n$  i.i.d. with p.d.f.
  - (a)  $f(x; \theta) = 1, \theta - 0.5 < x < \theta + 0.5; 0$ , otherwise.
  - (b)  $f(x; \theta) = 2x/\theta^2, (\theta > 0), 0 < x < \theta; 0$ , otherwise.
5. Let  $X_1, \dots, X_n$  be a random sample from a  $N(0, \sigma_x^2)$  and  $Y_1, \dots, Y_m$  be a random sample from  $N(0, \sigma_y^2)$ , independent of the  $X$ 's. Define  $\lambda = \sigma_y^2/\sigma_x^2$ . Find a  $1 - \alpha$  confidence interval for  $\lambda$  by inverting that LRT.

6. Let  $X_1, \dots, X_n$  be i.i.d. random variables from  $N(\theta, \sigma^2)$ , where  $\sigma^2$  is also unknown. As usual,  $\bar{X} = \sum X_i/n$  and  $S^2 = \sum (X_i - \bar{X})^2/(n-1)$ .
- (a) Show that the interval  $\{\theta : \theta \leq \bar{X} + S t_{n-1, \alpha}/\sqrt{n}\} = (-\infty, \bar{X} + S t_{n-1, \alpha}/\sqrt{n}]$  can be derived by inverting the acceptance region of an LRT.
  - (b) Show that the interval  $[\bar{X} \pm S t_{n-1, \alpha/2}/\sqrt{n}]$  can also be derived by inverting the acceptance region of an LRT.
  - (c) Show that the intervals in parts (a) and (b) are unbiased.
7. Let  $X_1, \dots, X_n$  be a random sample from  $N(\mu, \sigma^2)$ .
- (a) If  $\sigma^2$  is known, find a minimum sample size  $n$  to guarantee that the 95% two-sided UMAU confidence interval for  $\mu$  will have length no more than  $\sigma/4$ .
  - (b) If  $\sigma^2$  is unknown, how to find a minimum sample size  $n$  to guarantee, with probability 0.9, that the 95% two-sided UMAU confidence interval for  $\mu$  will have length no more than  $\sigma/4$ ?
8. Let  $X$  be a single observation from  $\text{Beta}(\theta, 1)$ ,  $\theta > 0$ .
- (a) Let  $Y = -(\ln X)^{-1}$ . Evaluate the confidence coefficient of the interval  $[Y/2, Y]$ , that is: calculate  $\inf_{\theta > 0} P_{\theta}(\theta \in [Y/2, Y])$ .
  - (b) Find a pivot-based confidence interval having the same confidence coefficient as the interval in part (a).