

1. let $(X, Y) \sim N(0, 0, \sigma_x^2, \sigma_y^2, \rho)$. Show that $X + Y$ and $X - Y$ are independent if and only if $\sigma_x = \sigma_y$.
2. Consider the general linear regression model : $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{\epsilon}$, where $E(\underline{\epsilon}) = \underline{0}$, $\sigma^2\{\underline{\epsilon}\} = \sigma^2 \cdot I_{n \times n}$, $\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_k)^t$, $p = k + 1 < n$.
 - (a) Show that \underline{b} is a least squares estimate of $\underline{\beta}$ if and only if \underline{b} satisfies the normal equations:

$$\mathbf{X}^t \underline{Y} = \mathbf{X}^t \mathbf{X} \underline{b}.$$
 - (b) Now, assume $\text{rank}(\mathbf{X})$ is p . Show that $\text{SSTO} = \underline{Y}^t P_1 \underline{Y}$, $\text{SSR} = \underline{Y}^t P_2 \underline{Y}$ and $\text{SSE} = \underline{Y}^t P_3 \underline{Y}$ with each P_j , $j = 1, 2, 3$ be a $n \times n$, symmetric and idempotent matrix. Find $\text{rank}(P_j)$, $j = 1, 2, 3$.
 - (c) If, furthermore, assume each ϵ_i , $i = 1, \dots, n$, distributes normally. Show the independence between SSR and SSE.
3. A student fitted a linear regression function for a class assignment. The student plotted the residuals e_i against Y_i and found a positive relation. When the residuals were plotted against the fitted values \hat{Y}_i , the student found no relation. How could the difference arise?
4. Consider the model: $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{\epsilon}$, where $E(\underline{\epsilon}) = \underline{0}$, $\sigma^2\{\underline{\epsilon}\} = \sigma^2 \cdot I_{n \times n}$, the $n \times p$ design matrix \mathbf{X} has rank p , $p < n$.
Now, consider the model : $\underline{Y}^* = \mathbf{X}^* \underline{\beta} + \underline{\epsilon}^*$, where $\underline{Y}^* = A\underline{Y}$, $\mathbf{X}^* = \mathbf{A}\mathbf{X}$, $\underline{\epsilon}^* = A\underline{\epsilon}$ and A is a known $n \times n$ orthogonal matrix.
Show that
 - (a) $E(\underline{\epsilon}^*) = \underline{0}$, $\sigma^2\{\underline{\epsilon}^*\} = \sigma^2 \cdot I_{n \times n}$
 - (b) $\underline{b} = \underline{b}^*$ and $\text{MSE} = \text{MSE}^*$,
where \underline{b} and \underline{b}^* are the least squares estimators of $\underline{\beta}$; and MSE and MSE^* are the unbiased estimators of σ^2 obtained from the two models, respectively.
5. Observation vector $\underline{Y} = (Y_1, Y_2, Y_3)^t$ has expected mean $\underline{\theta} = (0, \mu, 4\mu)^t$, where μ is a unknown parameter.
 - (a) Rewrite the case as in a linear regression model formulation: that is to find \mathbf{X} and $\underline{\beta}$ such that $E(\underline{Y}) = \mathbf{X}\underline{\beta}$.
 - (b) Let $\Omega = \{\underline{\theta} : \underline{\theta} = (0, \mu, 4\mu)^t, \mu \in R\}$. What is the space Ω here? Give the projection matrix H .
 - (c) Let $\underline{a} = (a_1, a_2, a_3)^t$ be any vector such that $\underline{a}^t \underline{Y}$ be a linear unbiased estimator for μ . Find the projection of \underline{a} onto Ω .
 - (d) Now, assume the the Gauss-Markov conditions hold for \underline{Y} , find the BLUE for μ .
 - (e) If, additionally, Y_1, Y_2, Y_3 are assumed independent normally distributed with common unknown variance σ^2 .
Show how to test the hypothesis $H_0 : \mu = 0$ v.s. $H_1 : \mu \neq 0$.