

1. Let X_1, \dots, X_n be a random sample from a $U(0, \theta)$, with unknown parameter $\theta > 0$, and let $Y = X_{(n)} = \max\{X_1, \dots, X_n\}$.
 - (a) Let $0 \leq c < d$ be given constant, and consider the random interval $C_1(Y) = [Y + c, Y + d]$. Calculate the coverage probability function of $C_1(Y)$, then show that the confidence coefficient of $C_1(Y)$ is 0.
 - (b) Let $1 \leq a < b$ be given constant, and consider the random interval $C_2(Y) = [aY, bY]$. Show that the coverage probability function of $C_2(Y)$ is a constant, hence equals to the confidence coefficient of $C_2(Y)$.
 - (c) Now, for a given $\alpha \in (0, 1)$, give three different pairs of a and b in (b) such that the confidence coefficient of $C_2(Y)$ is $1 - \alpha$.
2. Assume that α_1 and α_2 are any numbers such that $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Let U_α be an exact upper confidence bound for θ , i.e. $P_\theta(\theta \leq U_\alpha(X)) = \alpha, \forall \theta$. If $U_{\alpha_1}(x) \leq U_{\alpha_2}(x), \forall x$, prove that for all $\theta, P_\theta\{\theta \in [U_{\alpha_1}, U_{\alpha_2}]\} \geq \alpha_2 - \alpha_1$.
3. Let X_1, \dots, X_n be *i.i.d.* from $N(\theta, \sigma_0^2)$, with $\sigma_0 > 0$ and $\alpha \in (0, 1)$ are given constants; also let $X = (X_1, \dots, X_n)$.
 - (a) Consider the random lower confidence limit for θ :

$$C_1(X) = \left[\bar{X} - \frac{\sigma_0}{\sqrt{n}} Z_\alpha, \infty \right).$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of $C_1(X)$. Is $C_1(X)$ an unbiased confidence set for θ ?

- (b) Also, consider the random upper confidence limit for θ :

$$C_2(X) = \left(-\infty, \bar{X} + \frac{\sigma_0}{\sqrt{n}} Z_\alpha \right].$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of $C_2(X)$. Is $C_2(X)$ an unbiased confidence set for θ ?

- (c) Now, consider the random confidence interval for θ :

$$C_3(X) = \left[\bar{X} - \frac{\sigma_0}{\sqrt{n}} Z_{\alpha/2}, \bar{X} + \frac{\sigma_0}{\sqrt{n}} Z_{\alpha/2} \right].$$

Calculate the coverage probabilities and the (relevant) false coverage probabilities of $C_3(X)$. Is $C_3(X)$ an unbiased confidence set for θ ?

4. Find a $1 - \alpha$ confidence interval for θ , given X_1, \dots, X_n i.i.d. with p.d.f.
 - (a) $f(x; \theta) = 1, \theta - 0.5 < x < \theta + 0.5; 0$, otherwise.
 - (b) $f(x; \theta) = 2x/\theta^2, (\theta > 0), 0 < x < \theta; 0$, otherwise.
5. Let X_1, \dots, X_n be a random sample from a $N(0, \sigma_x^2)$ and Y_1, \dots, Y_m be a random sample from $N(0, \sigma_y^2)$, independent of the X 's. Define $\lambda = \sigma_y^2/\sigma_x^2$. Find a $1 - \alpha$ confidence interval for λ by inverting that LRT.

6. Let X_1, \dots, X_n be i.i.d. random variables from $N(\theta, \sigma^2)$, where σ^2 is also unknown. As usual, $\bar{X} = \sum X_i/n$ and $S^2 = \sum (X_i - \bar{X})^2/(n-1)$.
- (a) Show that the interval $\{\theta : \theta \leq \bar{X} + S t_{n-1, \alpha}/\sqrt{n}\} = (-\infty, \bar{X} + S t_{n-1, \alpha}/\sqrt{n}]$ can be derived by inverting the acceptance region of an LRT.
 - (b) Show that the interval $[\bar{X} \pm S t_{n-1, \alpha/2}/\sqrt{n}]$ can also be derived by inverting the acceptance region of an LRT.
 - (c) Show that the intervals in parts (a) and (b) are unbiased.
7. Let X_1, \dots, X_n be a random sample from $N(\mu, \sigma^2)$.
- (a) If σ^2 is known, find a minimum sample size n to guarantee that the 95% two-sided UMAU confidence interval for μ will have length no more than $\sigma/4$.
 - (b) If σ^2 is unknown, how to find a minimum sample size n to guarantee, with probability 0.9, that the 95% two-sided UMAU confidence interval for μ will have length no more than $\sigma/4$?
8. Let X be a single observation from $\text{Beta}(\theta, 1)$, $\theta > 0$.
- (a) Let $Y = -(\ln X)^{-1}$. Evaluate the confidence coefficient of the interval $[Y/2, Y]$, that is: calculate $\inf_{\theta > 0} P_{\theta}(\theta \in [Y/2, Y])$.
 - (b) Find a pivot-based confidence interval having the same confidence coefficient as the interval in part (a).