1. let $(X, Y) \sim N(0, 0, \sigma_x^2, \sigma_y^2, \rho)$. Show that $X + Y$ and $X - Y$ are independent if and only if $\sigma_x = \sigma_y$.

2. Consider the general linear regression model : $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{\epsilon}$,
   where $E(\underline{\epsilon}) = \underline{0}$, $\sigma^2\{\underline{\epsilon}\} = \sigma^2 \cdot I_{n\times n}$, $\underline{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)^t$, $p = k + 1 < n$.

   (a) Show that $\underline{b}$ is a least squares estimate of $\underline{\beta}$ if and only if $\underline{b}$ satisfies the normal equations:
   $$\mathbf{X^t}\underline{\mathbf{Y}} = \mathbf{X^t X}\underline{\mathbf{b}}.$$

   (b) Now, assume rank($\mathbf{X}$) is p. Show that SSTO= $\underline{Y}^t P_1 \underline{Y}$, SSR= $\underline{Y}^t P_2 \underline{Y}$ and SSE= $\underline{Y}^t P_3 \underline{Y}$ with each P_j, $j = 1,\ 2,\ 3$ be a $n \times n$, symmetric and idempotent matrix. Find rank(P_j), $j = 1,\ 2,\ 3$.

   (c) If, furthermore, assume each ϵ_i, $i = 1, \ldots, n$, distributes normally. Show the independence between SSR and SSE.

3. A student fitted a linear regression function for a class assignment. The student plotted the residuals e_i against Y_i and found a positive relation. When the residuals were plotted against the fitted values $\hat{Y}_i$, the student found no relation. How could the difference arise?

4. Consider the model: $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{\epsilon}$, where $E(\underline{\epsilon}) = \underline{0}$, $\sigma^2\{\underline{\epsilon}\} = \sigma^2 \cdot I_{n\times n}$, the $n \times p$ design matrix $\mathbf{X}$ has rank p, $p < n$.
   Now, consider the model : $\underline{Y}^* = \mathbf{X}^*\underline{\beta} + \underline{\epsilon}^*$, where $\underline{Y}^* = A\underline{Y}$, $\mathbf{X}^* = \mathbf{AX}$, $\underline{\epsilon}^* = A\underline{\epsilon}$ and A is a known $n \times n$ orthogonal matrix.
   Show that

   (a) $E(\underline{\epsilon}^*) = \underline{0}$, $\sigma^2\{\underline{\epsilon}^*\} = \sigma^2 \cdot I_{n\times n}$

   (b) $\underline{b} = \underline{b}^*$ and MSE=MSE*,
   where $\underline{b}$ and $\underline{b}^*$ are the least squares estimators of $\underline{\beta}$; and MSE and MSE* are the unbiased estimators of σ^2 obtained from the two models, respectively.

5. Observation vector $\underline{Y} = (Y_1, Y_2, Y_3)^t$ has expected mean $\underline{\theta} = (0, \mu, 4\mu)^t$, where μ is a unknown parameter.

   (a) Rewrite the case as in a linear regression model formulation: that is to find $\mathbf{X}$ and β such that $E(\underline{Y}) = \mathbf{X}\beta$.

   (b) Let $\Omega = \{\underline{\theta} : \underline{\theta} = (0, \mu, 4\mu)^t,\ \ \mu \in R\}$. What is the space Ω here? Give the projection matrix H.

   (c) Let $\underline{a} = (a_1, a_2, a_3)^t$ be any vector such that $\underline{a}^t\underline{Y}$ be a linear unbiased estimator for μ. Find the projection of $\underline{a}$ onto Ω.

   (d) Now, assume the the Gauss-Markov conditions hold for $\underline{Y}$, find the BLUE for μ.

   (e) If, additionally, Y_1, Y_2, Y_3 are assumed independent normally distributed with common unknown variance σ^2.
   Show how to test the hypothesis $H_0 : \mu = 0$ v.s. $H_1 : \mu \neq 0$.